Solid-State Physics Exercises HS 2023

1) Bravais lattice

Why does a tetragonal base-centered Bravais lattice not occur?

2) Cubic lattice

Give the following quantities for simple cubic, bcc, and fcc lattices of lattice constant *a:*

-Volume of the unit cell
-Number of primitive lattice points per unit cell
-Volume of the primitive unit cell
-number of nearest neighbors (so-called coordination number)
-nearest neighbor distance
-Packing density for contacting spherical atoms

3) Lattice constant of Au

Gold has a cubic fcc lattice and a density of 19.3 g/cm^3. Calculate the lattice constant, nearest neighbor distance and radius of a gold atom (assumption: touching spheres).

4) Wigner-Seitz cell

Construct the Wigner-Seitz cell of the orthorhombic base-centered lattice if $a_1 : a_2 : a_3 = 4:2:3$.

5) Sphere packings

Determine the ratio *c/a* of an ideal hexagonal close packed sphere packing (hcp). Is the packing density of a fcc sphere packing larger or smaller than that of a hcp sphere packing?

6) Structure amplitude

Calculate the structure amplitude *S* as a function of *hkl* for the NaCl structure assuming that the atomic scattering factors *f* for Na and Cl are each constant but different.

What would happen if the atoms had identical atomic scattering factors? Which real substance with NaCl structure corresponds most closely to this assumption?

7) Atomic scattering factor

Calculate the atomic scattering factor *f* for a homogeneously charged sphere of charge *Z* and radius *R* as a function of Δk. Plot *f* as a function of the scattering angle sin(Θ) if $\lambda = R$ is assumed.

N.B.: The atomic scattering factor for a given atom is generally given by its electron density distribution $n(\vec{r})$ (resp. the charge density distribution $\rho(\vec{r}) = \mathrm{e}\, n(\vec{r})$) according to

$$f(\Delta\vec{k}) = \iiint n(\vec{r}) \mathrm{e}^{\mathrm{i}\Delta\vec{k}\circ\vec{r}} d^3r .$$

Integrate over the volume of the sphere in appropriately suitable coordinates with *n* = *const.* (*n* can be calculated from *Z* and *R*).

8) Debye-Waller factor and zero-point oscillation

Compare the atomic scattering factor *f*(sin(Θ) (normalized to 1 for Θ = 0º) from task 7 for $\lambda = R$ in a graph with the Debye-Waller factors for *T* = 4.2 K and *T* = 300 K if Li atoms (*R* = 0.123 nm) with a vibration frequency of 5×10^{13} Hz are considered. How large is the effect of zero-point vibration of the atoms in comparison?

9) Reciprocal lattice vectors

a) Show that a reciprocal lattice vector $\vec{G}_{hkl} = h\vec{b}_1 + k\vec{b}_2 + l\vec{b}_3$ is always perpendicular to the lattice planes (*hkl*).

(b) (optional): Show that the distance d_{hkl} of two lattice planes with Miller's indices (*hkl*) is given by $d_{hkl} = \dfrac{2\pi N}{\left|h\vec{b}_1 + k\vec{b}_2 + l\vec{b}_3\right|}$. What does the number *N* mean here?

10) Reciprocal lattice

Determine the reciprocal lattice vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$

a) for cubic fcc lattices and cubic bcc lattices,

b) for the hexagonal primitive lattice.

11) Ewald construction

Using Ewald's construction, qualitatively discuss what type of interference patterns can be expected when monochromatic light is diffracted by a linear point or line lattice.

12) Brioullin zones in reciprocal lattice

For a two-dimensional, single-rectangular lattice with $\vec{a}_2 = 2\,\vec{a}_1$, construct the first four Brioullin zones.

13) Laue image

Estimate the maximum possible number of interference maxima of a Laue image when the voltage of the X-ray tube used is 60 kV and the crystal has a simple cubic lattice with a lattice constant of 0.2 nm (let the X-ray tube produce a continuous bremsstrahlung spectrum).

14) Binding energy

a) Show that for a potential of the form $U(R) = -\frac{A}{R^m} + \frac{B}{R^n}$ a stable equilibrium position is reached if and only if $n > m$.

b) For a pure van der Waals attraction, the potential is often written as

$$U(R) = 4\varepsilon\left[(\sigma/R)^{12} - (\sigma/R)^{6}\right].$$

What is the binding energy E_B and the equilibrium distance R_0?

c) Calculate the effect of thermal expansion, $\Delta R_0(T)/R_0$, on a linear chain of atoms with the potential of subtask b). To do this, assume that the thermal energy $k_B T << E_B$ allows the atoms to move about the equilibrium position and consider the limits within which the atom can oscillate. From this, determine the mean position and compare the result with R_0.

Note: Use the series expansion $1/(1 \pm \varepsilon) \approx 1 \mp \varepsilon + \varepsilon^2 + ...$ to 2nd order, and $\sqrt[n]{1+\varepsilon} \approx 1 + \varepsilon/n + ...$ for $\varepsilon \to 0$.

15) Madelung constant

Calculate the Madelung constant for an infinite linear ion chain with alternating singly positively and negatively charged ions with constant spacing.

16) Elastic waves in lattices and in continuous media

In continuous media, the 1-D wave equation is $\frac{\partial^2 \xi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \xi(x,t)}{\partial x^2}$, with speed of sound $v = \sqrt{E/\rho}$, elastic modulus E and density ρ. For a linear atomic chain with atomic distance *a,* masses *m,* and spring constants *C,* we obtained $m\frac{\partial^2 \xi_n}{\partial t^2} = C(\xi_{n+1} + \xi_{n-1} - 2\xi_n)$. Show that in the limiting case for continuous media $\lambda >> a$, this equation of motion transforms into the 1-D wave equation. What is E expressed by C, m and a?

17) Linear atomic chain with diatomic basis

Given a NaCl chain with lattice constant a = 0.56 nm (consider whether this is identical to the Na-Cl atomic distance!).

a) Estimate the spring constant C from the result in Task 16 if E = 40 GPa and the chain is initially taken to be monatomic with atomic spacing *a/2.*

b) What is the speed of sound for small frequencies? What is the maximum possible oscillation frequency? What is the width of the forbidden oscillation range (expressed in eV with $E = \hbar\omega$)?

18) Linear atomic chain with different spring constants

For a linear chain of identical atoms of mass *m,* calculate the dispersion relation $\omega(k)$, if the atomic distance is $d = a/2$ and the spring constants C_1 and C_2 alternate (so that the unit cell with 2 identical atoms has lattice constant a). Draw $\omega(k)$ for C_1 / C_2 = 1.0, 0.6, 0.3 and 0.1.

19) Acoustic and optical vibrations in 2D

Sketch longitudinal acoustic and optical (LA and LO) and transverse acoustic and optical (TA and TO) oscillations for a 2-dimensional NaCl structure of lattice constant a. Let the wave vector with $\lambda = 4a$ be in the [1 0] direction.

20) Density of states for continuous media

What is the density of states for natural oscillations in a linear continuous medium (e.g. long homogeneous rod) of length L if the speed of sound is constant? To do this, first determine the possible k values and consider what determines the upper limits for k or the vibration frequency ω in a real material.

Compare the result graphically with the density of states for a monatomic chain from the lecture.

21) Density of states with 2-atomic basis

In the lecture, the density of states $\rho(\omega)$ for natural oscillations of a diatomic chain was sketched without calculation. How exactly does $\rho(\omega)$ diverge for $k \to 0$ for the optical branch? Use the result from the lecture,

$$\omega_{\pm}^{\,2} = \frac{C}{\mu}\left(1 \pm \sqrt{1 - \frac{4\mu^2 \sin^2(ka/2)}{m_1 m_2}}\right) \quad \text{mit} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Use appropriate expansions for $\sqrt{1+x}$ and $\sin(x)$.

22) Inelastic scattering of light by phonons

In the lecture, an energy-momentum scheme was shown illustrating inelastic scattering of neutrons by phonons. Draw an analogous, qualitatively correct scheme for the inelastic scattering of *light* by phonons. Make sure that the ratio of phonon frequencies/pulses to the corresponding light frequencies/pulses is physically reasonable. What condition must a solid body fulfill for its phonons to absorb light particularly efficiently, and for what wavelengths of light does this typically occur?

23) Number of phonons in a solid body

How many phonons occupy a single lattice vibration of circular frequency ω_0 with i) $\omega_0 = 10^4$ s^{-1} and ii) $\omega_0 = 10^{13}$ s^{-1} , at $T = 0.1$ K, 1 K, 10 K, 100 K, 1000 K?

24) Debye's law in 1D and 2D

a) Derive an expression for the phonon density of states $\rho(\omega)$ for a 1D chain and a 2D lattice with N identical atoms at given constant sound velocities c and particle densities N/L or N/L^2 for periodic boundary conditions (1D: only longitudinal oscillations; 2D: only one longitudinal and one transverse oscillation). What is the maximum oscillation frequency in each case?

b) How do the specific heats for the 1D chain and the 2D lattice at low temperatures depend qualitatively on T (without calculating the prefactors)? Use the expression from the lecture for the mean energy of a natural oscillation,

$$\overline{E}_\omega = \frac{\hbar\omega}{\exp(\hbar\omega/\mathrm{k_B}T) - 1}.$$

25) Debye temperature of solid argon

Determine from the measured data of the specific heat of argon (see Fig 1) the Debye temperature Θ_D resp. the Debye frequency ω_D. The density of solid argon is 1.8 g/cm^3. What are the speed of sound and the elastic modulus? Why is Θ_D comparatively small compared to other solids?

26) Einstein model of specific heat

In the Einstein model of the specific heat of a solid, only one possible oscillation frequency ω_E is assumed. What prefactor ρ_0 must the corresponding density of states $\rho(\omega) = \rho_0\delta(\omega - \omega_E)$ have? What would be a reasonable estimate for ω_E if ω_D is given? Sketch $C_v(T)$ with this estimate, in comparison with the specific heat from Debye theory (see Fig. 2).

27) Thermal conductivity of LiF

From the material properties of LiF, determine the thermal conductivity as a function of temperature for $T \to 0$ and compare the result for a crystal size of 1.06 mm with the literature values (see Fig. 3).

28) 1D and 2D metals

Determine the electronic densities of states $D(E)$ and corresponding expressions for the Fermi energy for one- and two-dimensional free electron gases with electron densities $n = N/L$ *and* $n = N/L^2$, respectively.

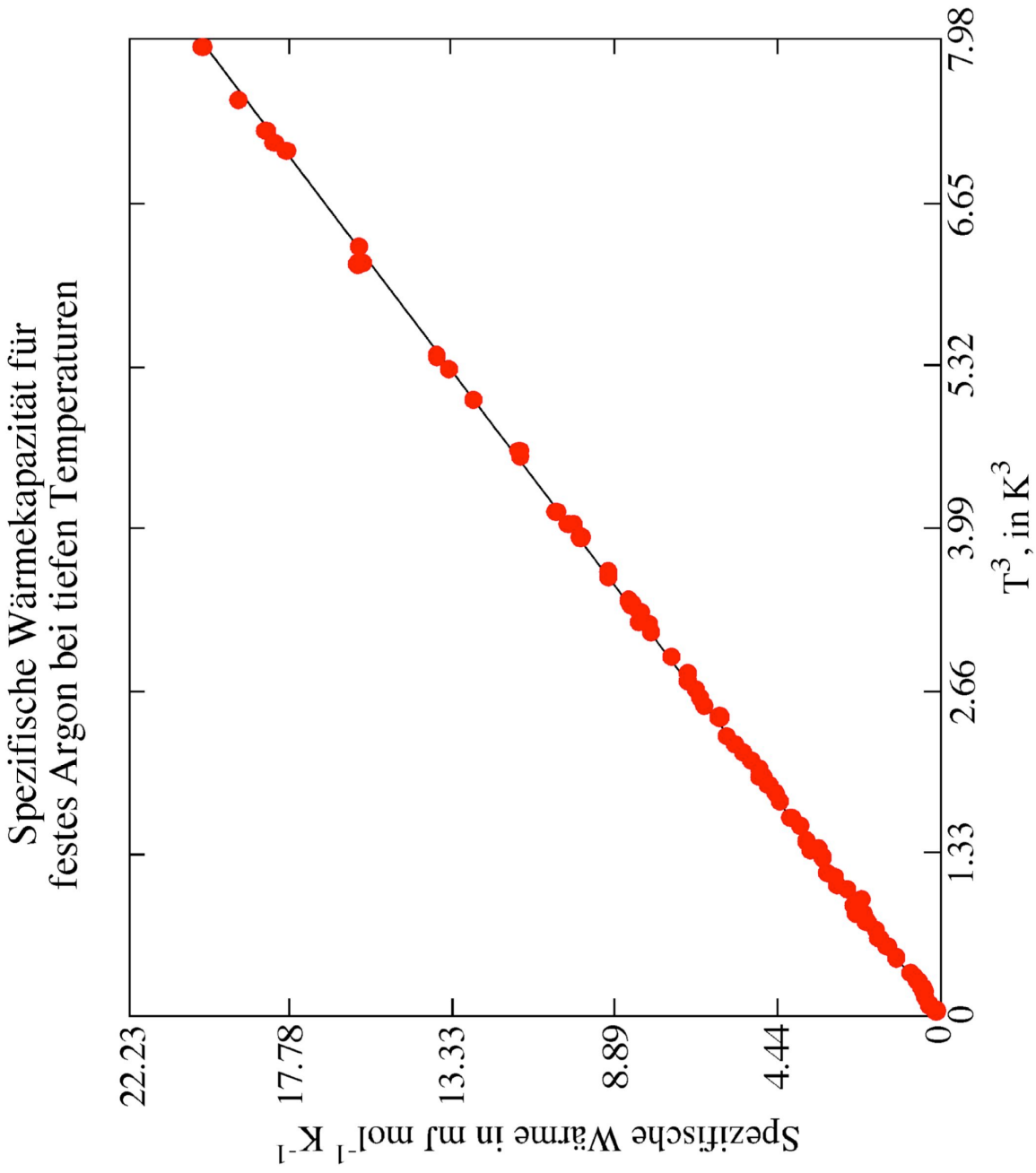


Fig. 1: Measured data of the specific heat of argon

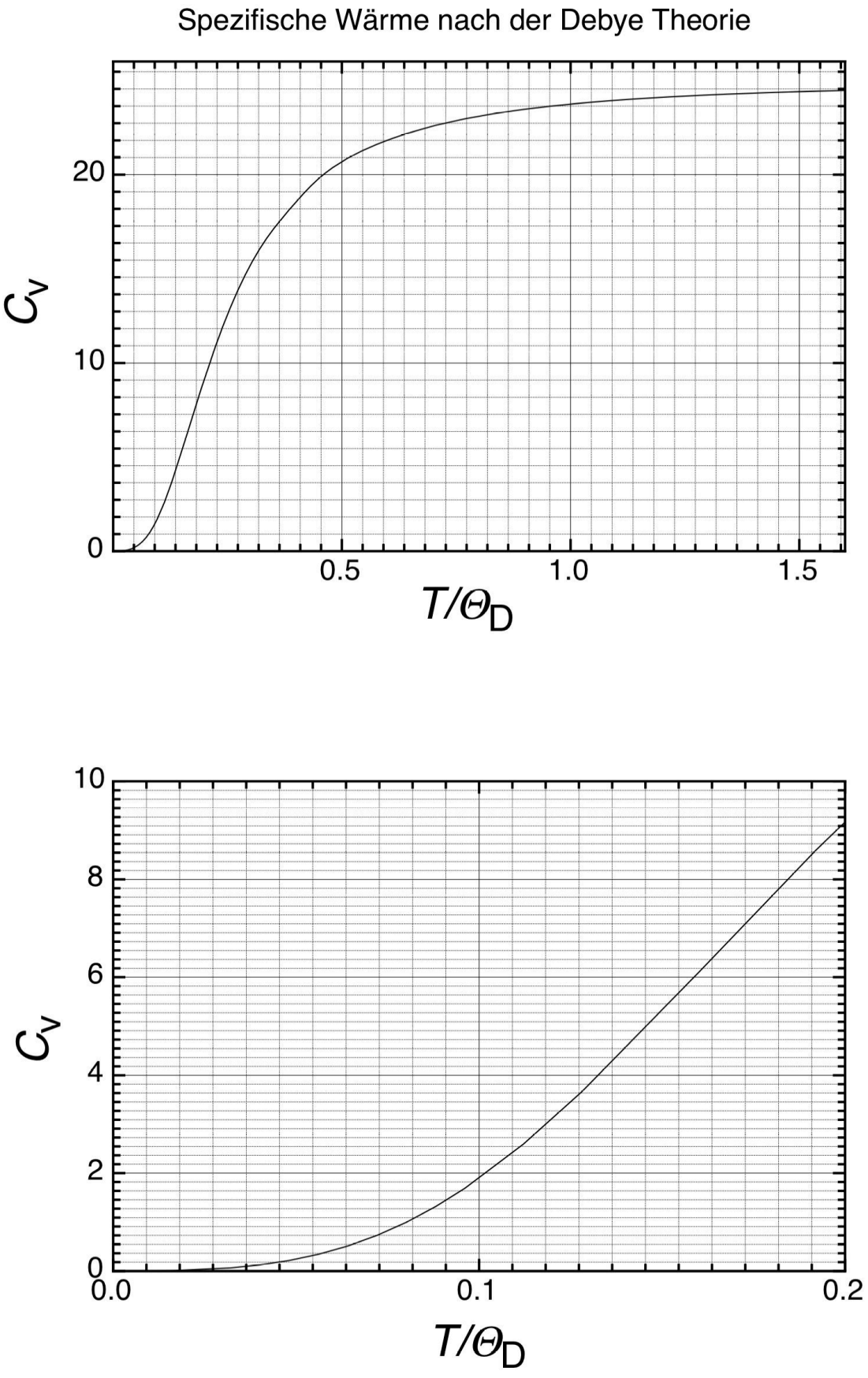


Fig. 2: Specific heat from the Debye theory

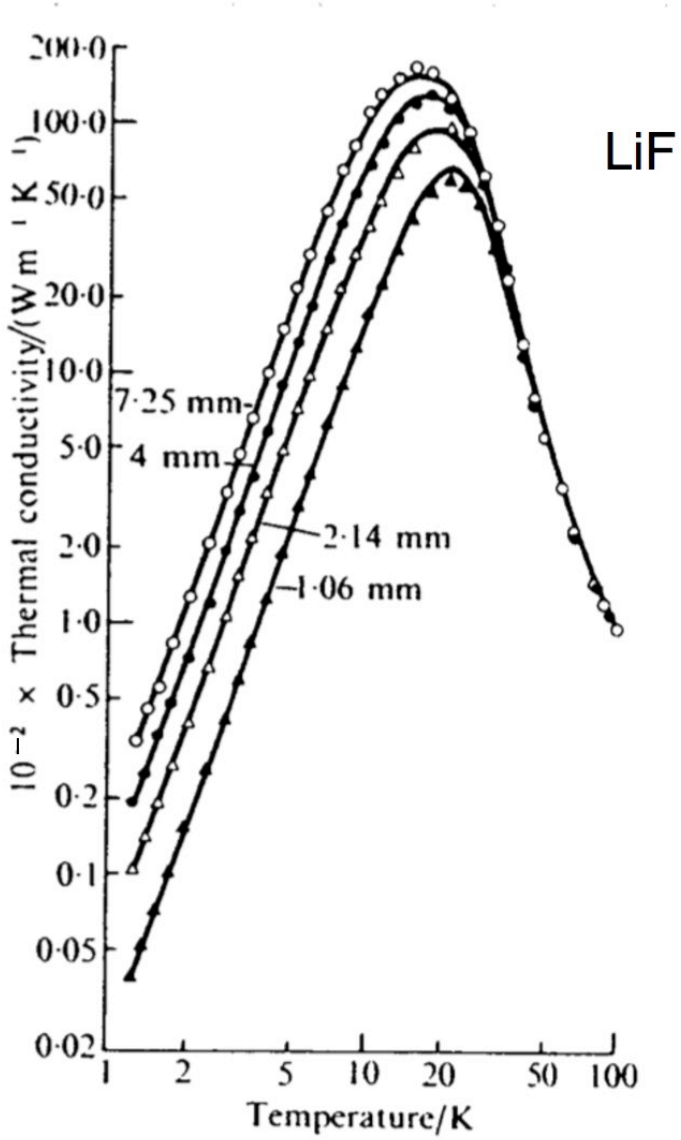


Fig. 3: Material properties of LiF

Chemical Formula	LiF
Crystal Class	Cubic
Lattice Constant, Å	4.03
Molecular Weight	25.94
Density, g/cm^3 (20 °C)	2.60
Reflection Loss, % for two surfaces at 4 μm	4.4
Dielectric Constant for 10^2-10^{10} Hz at 298 K	9.0
Melting Temperature, K	1140
Thermal Conductivity, W/(m K) at 314 K	11.3
Thermal Expansion, 1/K at 300 K	34.4 x 10^{-6}
Specific Heat, cal/(g K) at 283 K	0.37
Debye Temperature, K	732
Bandgap, eV	13.6
Solubility, g/100 g H_2O at 291 K	0.27
Knoop Hardness, kg/mm^2	100
Young's Modulus, GPa	64.77
Apparent Elastic Limit, MPa	11.2
Shear Modulus, GPa	55.12
Bulk Modulus, GPa	62.0
Poisson Ratio	0.326

29) Fermi gases Na, ^{3}He, neutrons

Calculate the Fermi energy (in eV), Fermi temperature, and Fermi velocity of a) sodium metal ($\rho = 0.97\text{g/cm}^3$, m_{mol} = 23 g/mol), b) liquid ^{3}He (density $\rho = 82$ mg/cm^3 , m_{mol} = 3 g/mol), and c) a neutron star with 1.5 solar masses and 15 km radius (without relativistic effects).

30) Mean energy, pressure and compressibility of the electron gas.

a) Determine the average energy of the electrons in the free electron gas at $T = 0$, expressed by the Fermi energy.

b) Determine the electron pressure in the free electron gas at $T = 0$, where you can use that $p = -\left(\partial U / \partial V\right)_N$ (First, convince yourself that this relation from thermodynamics is satisfied at all for ideal gases at $T > 0$).

(c) Determine the volume compressibility ε of the free electron gas at $T = 0$ and expressed by p, where $\varepsilon = -(1/V)\left(\partial V / \partial p\right)_N$.

d) Calculate the electron pressure and compressibility of Li metal.
($\rho = 0.534\text{g/cm}^3$, m_{mol} = 6.94 g/mol), and compare ε with the measured compressibility of $\varepsilon \approx 9\text{x}10^{-11}$ Pa^{-1}.

31) Chemical potential for a 2D electron gas

Calculate the chemical potential $\mu(T)$ for a two-dimensional gas of electrons with electron surface density $n = N/L^2$. Sketch $\mu(T)$ and compare the limiting case $T = 0$ with the result for the Fermi limit energy from task 28.

32) Chemical potential at discrete states

A system which can only assume the (non-degenerate) discrete energy states E_1 and $E_2 > E_1$ is occupied by fermions.

a) What is the density of states for this system?

b) What is the chemical potential μ for $T = 0$, if exactly one fermion occupies the system? What can be said about μ for $T = 0$ if exactly two fermions occupy the system?

c) How large is $\mu(T)$ for two fermions and $T > 0$?

33) Specific heat of copper

Try to determine the electron mass from the available measured data for copper (see Table 1, $\rho = 8.94\,\mathrm{g/cm^3}$, m_{mol} = 63.55 g/mol). By how much does the value obtained deviate from the literature value for free electrons?

T (K)	C_V (mJ mol^{-1}K^{-1}
0.25	0.17
0.50	0.35
0.75	0.54
1.00	0.74
1.25	0.96
1.50	1.21
1.75	1.47
2.00	1.78
2.25	2.11
2.50	2.50
2.75	2.91
3.00	3.35
3.25	3.91
3.50	4.46
3.75	5.15
4.00	5.87
4.50	7.49

Table 1: Spezische Wärme von Kupfer.

34) Drift velocity, Fermi velocity and residual resistance in copper.

a) What is the drift velocity of the conduction electrons in copper (density $\rho = 8.94\,\mathrm{g/cm^3}$, m_{mol} = 63.55 g/mol, one conduction electron per atom) when a current of 10 A is passed through a wire of 1 mm diameter? How large is the Fermi velocity in comparison?

b) What is the mean free path in copper at room temperature with a resistivity of $1.7\mathrm{x}10^{-8}$ Ωm?

c) Copper with 3.3 atomic percent nickel as impurity has a residual resistance of approx. $4\mathrm{x}10^{-8}$ Ωm. What is the mean distance of the Ni atoms compared to the mean free path of the electrons at $T = 0$?

35) Hall effect

What is the maximum measurable Hall voltage in a 10 cm long, 0.1 mm thin and 1 mm wide copper foil when an applied current of 1 A flows along the longitudinal axis and the available magnetic field is 5 T? How exactly does this voltage depend on the geometry of the foil for a given magnetic field and measuring current?

How does the Hall voltage change when the foil is continuously rotated about its 3 axes of symmetry by 90 degrees each with the current and voltage contacts firmly soldered on?

36) Faraday balance for measuring magnetic susceptibility

What is the dimensionless (volume) Pauli spin susceptibility of copper, $\chi_{Pauli} = M/H$? A small piece of copper is placed in an inhomogeneous magnetic field with a constant field gradient dB/dz = 10 T/m and an average magnetic field of B = 2 T. How large is the force due to the free electrons compared to the weight of the copper? *Reminder:* The magnetic field energy density is $BH/2$, the force is the gradient of the potential energy.

37) Trajectories in the magnetic field

a) What is the classical orbital radius of electrons at the Fermi edge for copper in a magnetic field of B = 5 T (ρ = 8.94g/cm^3, m_{mol} = 63.55 g/mol, 1 conduction electron per atom)? What is the corresponding cyclotron angular frequency?

b) How large may the residual resistance be, so that on average just one complete undisturbed circulation is possible without impacts of impurities?

38) De Haas-van Alphen effect in gold

Estimate the Fermi energy of gold (in eV) based on the oscillation of the Pauli spin susceptibility in the magnetic field (see Fig. 4; *note:* the oscillation with the smaller period corresponds to the largest extreme orbit and can be assumed as the circumference of a Fermi sphere). Compare the result with the literature value E_F = 5.51 eV. Where must the periodicity with the longer period come from?

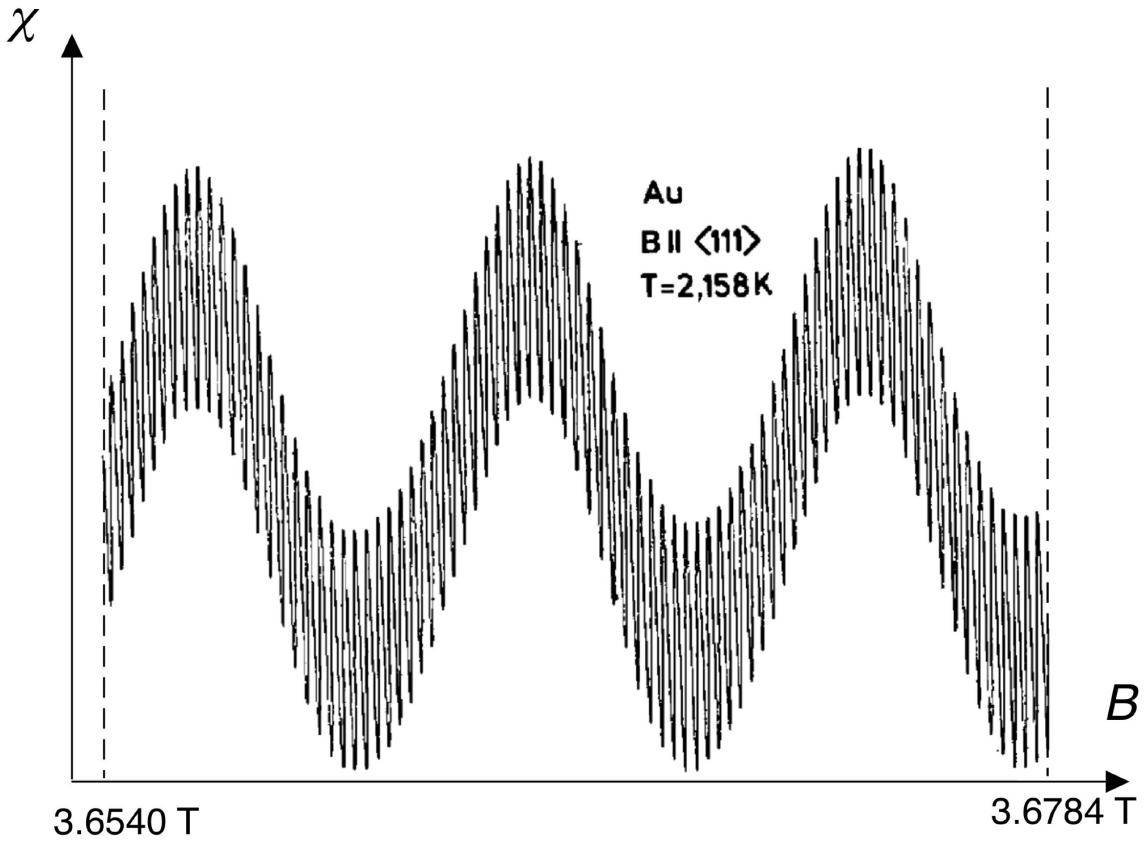
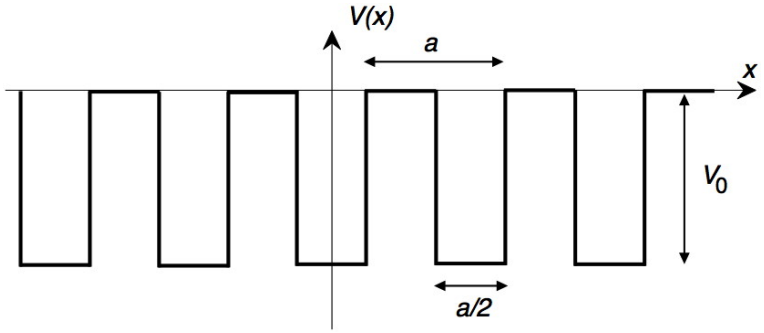


Fig. 4: Pauli spin susceptibility of gold in a magnetic field

39) Energy gaps for periodic potentials

We consider as a model a periodic 1D box potential of box width *a/2* and depth V_0 with box spacing *a* (simplified Kronig-Penney model).



How large are the energy gaps $E_{g,n}$, which separate the energy bands *n* and *n*+1?

40) Fermi line and reduced zone scheme in 2D

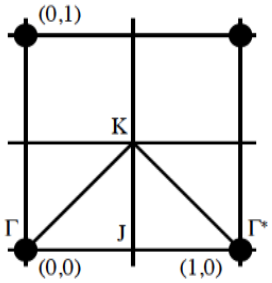
a) A cubic lattice of lattice constant *a in* two dimensions consists of atoms, each contributing 2 electrons. What is the magnitude of the Fermi wave vector k_F in units of π/a?

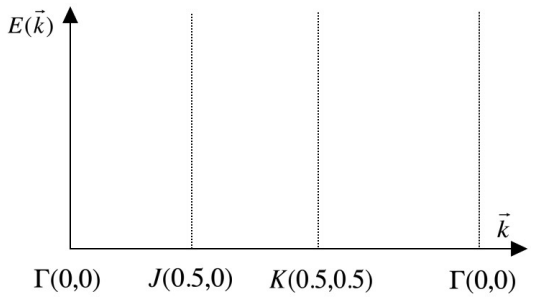
b) For this lattice, sketch what each of the reduced and the periodic zone schemes look like. Make separate sketches for free electrons and for electrons that are in a weak periodic potential.

41) Free electrons in the square lattice

We consider in the reciprocal lattice of a square lattice with lattice constants *a,* the wave vectors, $\vec{k}_\Gamma = 2\pi/a$ (0,0) $\vec{k}_J = 2\pi/a$ (0.5,0) and $\vec{k}_K = 2\pi/a$ (0.5,0.5) . Sketch $E(\vec{k})$ for the lowest two energy bands of free electrons along the line $\Gamma - J - K - \Gamma$. How would an analogous sketch look like for the line $\Gamma^* - J - K - \Gamma^*$ with $\vec{k}_{\Gamma^*} = 2\pi/a$ (1,0)?

Optional: What changes in a weak periodic potential?





42) Effective mass at cyclotron resonance

The average effective mass m^* for the conduction electrons in the semiconductor InSb is 0.014 m_e. How large must a magnetic field be for the conduction electrons to be excited to cyclotron resonance by microwaves with a wavelength of 3 cm?

43) Band overlap in semimetals

Let the overlapping energy bands of a semimetal near $E_1(0)$ or $E_2(k_0)$ be given as follows:

$$E_1(k) = E_1(0) - \frac{\hbar^2 k^2}{2m_1^*} \qquad \text{and} \qquad E_2(k) = E_2(k_0) + \frac{\hbar^2 (k-k_0)^2}{2m_2^*}.$$

a) Sketch $E(k)$ for $m_1^* > m_2^*$ and $E_1(0) > E_2(k_0)$ for a $k_0 > 0$ for which $E_1(k)$ and $E_2(k)$ do not intersect.

b) We assume that the material would be an insulator or semiconductor if the band $E_2(k)$ were not present at all. What can you say about the filling of the band $E_1(k)$ under this assumption? Sketch how the bands must be filled at $T = 0$ if both energy bands are present.

c) Which condition must apply to the number of holes or electrons in the two bands?

d) From this, calculate where the Fermi energy E_F must be for $T = 0$ by looking at the densities of states for both bands and relating these to the number of holes or electrons.

44) Bloch oscillations

Let an energy band be given in 1D by the expression $E(k) = \frac{\Delta}{2}\left[1 - \cos(ka)\right]$.

a) Sketch $E(k)$ and calculate the effective masses at $k = 0$ and $k = \pi/a$ in multiples of the electron mass for $\Delta = 1$ eV and $a = 0.3$ nm.

b) What is the equation of motion for $k(t)$ in a constant electric field E_0, and how does the energy of the electrons change as a function of time?

c) How does the group velocity change as a function of time, and what type of motion in real space does this correspond to? Compare this result for $E_0 = 1000$ V/m with the mean free path of Cu ($l \approx 3$ µm at $T = 300$ K). What is the condition for l so that the solution found can be realized at all?

45) Magnetization curves

In the lecture, we calculated the magnetization for a single electron which alone occupies an electron state, and we obtained the value $\mu_B \tanh(\mu_B B/k_B T)$ per electron. For general values for the total angular momentum *J*, the corresponding relationship is

$$g\mu_B J \left\{\frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J} g\mu_B JB/k_B T\right) - \frac{1}{2J}\coth\left(\frac{1}{2J} g\mu_B JB/k_B T\right)\right\}.$$

a) Show that the result obtained in the lecture is nevertheless correct.

b) By what factor does the pre-factor *C* of Curie's law *C/T* for high temperatures and small magnetic fields deviate from the value that we obtained for a single electron? *Hint: Use the expansion coth(x) ≈ 1/x + x/3 for small values of x.*

46) Demagnetization factor

In experiments to measure the magnetic susceptibility $\chi = M/H$ we refer to H_{ext} as the external magnetic field actually applied in the experiment, as you would calculate it from formulae for current-carrying magnetic coils, for example. Due to the induced magnetization *M* in the investigated sample of finite size, a magnetic dipole field is created around this sample as a first approximation, which weakens the external magnetic field by -*NM*. *N* is called the "demagnetization factor", which is geometry-dependent (see table below).

In other words, the sample "senses" an "effective" magnetic field $H_{eff} = H_{ext} - NM$. However, a measuring apparatus measures the actual magnetization *M*, regardless of how it is generated. The measured apparent susceptibility $\chi^* = M/H_{ext}$ then naturally deviates from the true magnetic susceptibility $\chi = M/H_{eff}$.

a) Find a relationship how you can obtain χ from χ^* if the demagnetization factor *N* is known.

b) What approximate value χ^* would you obtain if you measured a platelet-shaped crystal of a high-temperature superconductor in the superconducting state, which has a thickness of ≈ 0.2 mm and a diameter of 2 mm, and the magnetic field is applied perpendicular to the plane of the platelet?

Note: In the table below you will find values of N for cylinders with the ratio γ = length/diameter („N_m“) or for rotational ellipsoids with the ratio of the semi-axes a = b to c, γ =c/a („N”), where the magnetic field is parallel to the axis of symmetry in each case.

EXACT FLUXMETRIC AND MAGNETOMETRIC DEMAGNETIZING FACTORS N_f AND N_m FOR $\chi = 0^*$

γ	$N_m(0)$	$N_f(0)$	N
0.00001	0.9999	0.9999	1.0000
0.0001	0.9994	0.9993	0.9998
0.001	0.9950	0.9949	0.9984
0.01	0.9650	0.9638	0.9845
0.02	0.9389	0.9364	0.9694
0.03	0.9161	0.9124	0.9546
0.04	0.8954	0.8905	0.9402
0.05	0.8764	0.8703	0.9262
0.06	0.8586	0.8513	0.9125
0.07	0.8419	0.8333	0.8991
0.08	0.8261	0.8163	0.8860
0.09	0.8110	0.8001	0.8733
0.10	0.7967	0.7845	0.8608
0.12	0.7698	0.7553	0.8367
0.14	0.7450	0.7281	0.8137
0.16	0.7219	0.7027	0.7917
0.18	0.7004	0.6789	0.7706
0.20	0.6802	0.6565	0.7505
0.22	0.6611	0.6352	0.7312
0.24	0.6432	0.6151	0.7126
0.26	0.6262	0.5960	0.6948
0.28	0.6101	0.5778	0.6778
0.30	0.5947	0.5604	0.6614
0.32	0.5801	0.5438	0.6456
0.34	0.5662	0.5279	0.6304
0.36	0.5530	0.5127	0.6158
0.38	0.5403	0.4982	0.6017
0.40	0.5281	0.4842	0.5882
0.45	0.4999	0.4516	0.5563
0.50	0.4745	0.4221	0.5272
0.55	0.4514	0.3952	0.5005
0.60	0.4303	0.3705	0.4758
0.65	0.4110	0.3480	0.4531
0.70	0.3933	0.3273	0.4321
0.75	0.3770	0.3082	0.4126
0.80	0.3619	0.2905	0.3944
0.90	0.3349	0.2592	0.3618
1.0	0.3116	0.2322	0.3333
1.1	0.2911	0.2089	0.3083
1.2	0.2731	0.1886	0.2861
1.3	0.2572	0.1710	0.2664
1.4	0.2429	0.1555	0.2488
1.6	0.2186	0.1298	0.2187
1.8	0.1986	0.1096	0.1941
2.0	0.1819	0.09351	0.1736
2.5	0.1501	0.06544	0.1351
3.0	0.1278	0.04799	0.1087
3.5	0.1112	0.03653	0.08965
4	0.09835	0.02865	0.07541
5	0.07991	0.01889	0.05582
6	0.06728	0.01334	0.04323
7	0.05809	0.009904	0.03461
8	0.05110	0.007635	0.02842
9	0.04562	0.006061	0.02382
10	0.04119	0.004927	0.02029
20	0.02091	0.001245	0.006749
50	0.008438	0.0001999	0.001443
100	0.004232	0.00004999	0.0004299
200	0.002119	0.00001250	0.0001248
500	0.0008483	0.00000200	0.00002363
1000	0.0004243	0.00000050	0.00000661