Master Thesis

Landau-Ginzburg String Backgrounds with Orientifolds and D-branes

Author: Samo Jordan
Supervisor: Prof. Ilka Brunner
Institution: University of Zurich
(this work was done externally at the ETH Zurich)

May 28, 2008

Abstract

String backgrounds are usually described by superconformal field theories on the worldsheet. For a subset of these backgrounds an alternative description exists in terms of $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg theories. D-branes are realized as matrix factorizations of the Landau-Ginzburg superpotential and fit into the structure of a triangulated category. Parities can be defined as functors in these categories. In this work we use all these established results to give detailed instructions on how to explicitly construct string backgrounds with Orientifolds and D-branes, which solve the tadpole constraint and which are spacetime-supersymmetric. By including the most general permutation branes, many backgrounds can be constructed which currently do not have a CFT description.

Contents

1 Introduction

String theory in ten dimensions has developed itself into a candidate for a consistent theory of quantum gravity coupled to matter. The additional six dimensions are not observed in our macroscopic world, thus the idea of compactified dimensions was introduced, an idea originally going back to Kaluza and Klein. Much research has been devoted to study string compactification, first in the context of heterotic string theory. A good review on string compactification can be found in [1].

The second superstring revolution lead to the discovery of powerful dualities between different string theories, each of them being a different limit of a conjectured unified theory, commonly referred to as M-theory. D-branes have been recognized as being fundamental dynamical entities, whose solitonic nature makes them ideal candidates to study nonperturbative aspects of string theory. Moreover, since the D-branes carry non-abelian gauge fields, they can be used to build realistic type II string backgrounds.

The so-called geometrical compactifications use Calabi-Yau (CY) manifolds as internal target spaces. The inclusion of D-branes in compact CY spaces potentially spoils the consistency of the theory. This is due to the fact that D-branes carry Ramond-Ramond-charge [2]; a nonzero total charge leads to a tadpole anomaly [3]. To render the theory consistent, one needs to add further objects which compensate the RR-charge of the ones already present. The only candidates known are D-branes and Orientifolds.

String theories are perturbatively defined by specifying a superconformal field theory (SCFT) living on the worldsheet. The amount of worldsheet supersymmetry has a strong impact on the behaviour of the theory. Many supercharges help doing explicit calculations, but lead to a rigid structure of the theory, rendering it uninteresting. Models with little supersymmetry on the other hand are less calculable. The $\mathcal{N}=2$ case has become popular as a good compromise between both extremes [4].

Only a subset of all $\mathcal{N}=2$ SCFT's describe geometric compactifications. Well-known examples are nonlinear sigma models, where the target space coordinates are scalar fields living on the worldsheet. The non-geometric compactifications on the other hand describe models where the classical notions of geometry break down. The most simple $\mathcal{N}=2$ SCFT's belong to this class, namely the Minimal models [2]. These are rational conformal field theories (RCFT), which are exactly solvable. However, they can not be used as consistent string backgrounds, because the central charge does not have the right value ($c=9$). This problem can be overcome by tensoring together several Minimal models. The resulting $c=9$ theories are in general not spacetime-supersymmetric. Applying a certain orbifolding procedure yields the Gepner models, which describe exactly solvable non-geometric string backgrounds with spacetime-supersymmetry [5].

Some $\mathcal{N} = 2$ SCFT's can be described using $\mathcal{N} = 2$ Landau-Ginzburg (LG) theories. These are $\mathcal{N} = 2$ supersymmetric field theories which are not conformally invariant, but which flow into the SCFT under the renormalization group [4]. These LG theories are characterized by a quasihomogeneous superpotential $W(x_i)$, which determines the chiral ring structure of the SCFT. Gepner models are examples of SCFT's which admit a description in terms of LG orbifolds.

In a remarkable development, it has been shown that there exists a connection between Gepner models, characterized by the LG-superpotential $W(x_i)$, and nonlinear sigma models on CY target manifolds defined by the equation $W = 0$ in weighted projective space [6]. A more detailed analysis revealed that the nonlinear sigma model and the Landau-Ginzburg theory describe two different points in the same moduli space [7]. This connection is known as the LG/CY-correspondence.

$\mathcal{N} = 2$ supersymmetric theories can be twisted into topological field theories. This can be done in two different ways, leading to the A-model and the B-model, which can be interpreted as topological sectors of the untwisted theories. These two models are related to eachother by Mirror symmetry. D-branes and Orientifolds are accordingly called A-type resp. B-type, depending on whether the theory is A-twisted or B-twisted [8]. B-twisted Landau-Ginzburg theories are especially suitable to study topological string backgrounds, because correlation functions are easily computed using the formulas developed in [9, 10, 11].

Topological D-branes fit into the structure of a triangulated category [12]. In the case of B-twisted Landau-Ginzburg models, it is the category of matrix factorisations [13]. Orientifolds can be implemented in terms of functors in these categories [11]. The RR-charges of D-branes and Orientifolds can be computed using the formulas given in [11, 14].

In this work we use all these established results to give detailed instructions on how to construct tadpolefree and spacetime-supersymmetric string backgrounds with Orientifolds and D-branes in the Landau-Ginzburg language. We will be mostly interested in LG models of Fermat-type in five variables. In those cases where the LG theories flow into Gepner models, we compare our results with those of CFT calculations performed in [15].

2 $\mathcal{N} = 2$ Superconformal Field Theories

2.1 The $\mathcal{N} = 2$ superconformal algebra

We start by reviewing some elementary facts about $\mathcal{N} = 2$ SCFT's, following [4]. The holomorphic (left-moving) part of the $\mathcal{N} = 2$ superconformal algebra is generated by the energy momentum tensor $T(z)$, two supercharges $G^{\pm}(z)$ and a $U(1)$ current $J(z)$. They have the following mode expansions:

$$\begin{aligned} T(z) &= \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \\ G^{\pm}(z) &= \sum_{n=-\infty}^{\infty} G^{\pm}_{n\pm a} z^{-(n\pm a)-3/2} \\ J(z) &= \sum_{n=-\infty}^{\infty} J_n z^{-n-1} \end{aligned} \tag{1}$$

The real parameter a labels different $\mathcal{N} = 2$ SCFT's. There are two special cases: theories with $a = 0$ are said to be in the *Ramond sector*, while those with $a = 1/2$ are said to be in the *Neveu-Schwarz* sector. The holomorphic part of the $\mathcal{N} = 2$ superconformal algebra is now given by:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ [J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \\ [L_n, J_m] &= -mJ_{m+n} \\ [L_n, G^{\pm}_{m\pm a}] &= \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a} \\ [J_n, G^{\pm}_{m\pm a}] &= \pm G^{\pm}_{m+n\pm a} \\ \{G^{+}_{n+a}, G^{-}_{m-a}\} &= 2L_{m+n} + (n-m+2a)J_{n+m} + \frac{c}{3}\left[(n+a)^2 - \frac{1}{4}\right]\delta_{m+n,0} \end{aligned}$$

Note that the complete theory also contains the antiholomorphic (right-moving) sector with generators $\widetilde{T}(\bar{z})$, $\widetilde{G}^{\pm}(\bar{z})$ and $\widetilde{J}(\bar{z})$, satisfying the same relations like their holomorphic counterparts.

Let $\phi(z)$ be a primary field. By the state-operator correspondence we can associate a heighest weight state $|\phi\rangle$ to the primary field: $|\phi\rangle = \phi(0)|0\rangle$. It satisfies the following relations:

$$G_r^{\pm}|\phi\rangle = 0, \quad r \geq \frac{1}{2}, \qquad\qquad G^{\pm}_{-\frac{1}{2}}|\phi\rangle = |\Lambda^{\pm}\rangle \tag{2}$$

$$L_n|\phi\rangle = 0, \quad n \geq 1, \qquad\qquad L_0|\phi\rangle = h_\phi|\phi\rangle \tag{3}$$

$$J_n|\phi\rangle = 0, \quad n \geq 1, \qquad\qquad J_0|\phi\rangle = q_\phi|\phi\rangle \tag{4}$$

Here $|\Lambda\rangle$ denotes the superpartner of $|\phi\rangle$. Similar relations hold in the antiholomorphic sector. The theories with central charge $c = 3k/(k+2)$, $k = 1, 2, \dots$ are called $\mathcal{N} = 2$ *Minimal models*. They have only finitely many highest weight irreducible representations, which are labeled by the weight h_ϕ and by the $U(1)$-charge q_ϕ.

2.2 The chiral ring

A primary field is called a *chiral primary field*, if the associated state is annihilated by the operator $G^+_{-1/2}$. Similarly the state associated to a *antichiral primary field* is annihilated by the operator $G^-_{-1/2}$. In the antiholomorphic sector we define chiral and antichiral primary fields by replacing $G^{\pm}$ by $\widetilde{G}^{\pm}$. Thus we get four special types of NSNS primary fields, which we label with (c, c), (a, c), (a, a) and (c, a). Here the first letter refers to the holomorphic (left-moving) sector. These fields have interesting properties as we will see in the following.

From the SCFT algebra we extract the following anticommutators

$$\{G^-_{1/2}, G^+_{-1/2}\} = 2L_0 - J_0$$
$$\{G^-_{-1/2}, G^+_{1/2}\} = 2L_0 + J_0$$

which we sandwich between heighest weight states:

$$\langle\phi|\{G^-_{1/2}, G^+_{-1/2}\}|\phi\rangle = \langle\phi|2L_0 - J_0|\phi\rangle \tag{5}$$

$$\langle\phi|\{G^-_{-1/2}, G^+_{1/2}\}|\phi\rangle = \langle\phi|2L_0 + J_0|\phi\rangle \tag{6}$$

Unitarity implies $G^-_{1/2} = G^{+\dagger}_{-1/2}$ and $G^-_{-1/2} = G^{+\dagger}_{1/2}$, thus the left hand side of (5) and (6) is positive. Therefore all states in unitary $\mathcal{N} = 2$ SCFT's satisfy the following inequality:

$$h_\phi \geq \frac{1}{2}|q_\phi|$$

Furthermore (5) and (6) imply (using (2)) that the chiral resp. antichiral primaries saturate this bound:

$$h_\phi = +\frac{1}{2}q_\phi \qquad \text{(chiral primaries)} \tag{7}$$

$$h_\phi = -\frac{1}{2}q_\phi \qquad \text{(antichiral primaries)} \tag{8}$$

The same relations hold in the antiholomorphic sector. Furthermore it can be shown that all fields saturating the inequality are chiral resp. antichiral primaries.

Another important property of chiral primaries comes from the analysis of their operator products. It turns out that these are nonsingular and that the product of two chiral primaries is again a chiral primary. Thus the chiral primaries form a ring called the *chiral ring*. Similarly there is the *antichiral ring* consisting of antichiral primaries. By combining the holomorphic and the antiholomorphic sector we get a total of four rings, again denoted by (c,c), (a,c), (a,a) and (c,a). The latter two are complex conjugates of the first two. In the case of Minimal models and their tensor products these rings are finite, which is very helpful for doing explicit calculations. The importance of these four rings comes from the fact that they encode essential data of the SCFT.

2.3 Spectral flow

In every $\mathcal{N}=2$ superconformal theory there exist *spectral flow* operators $\mathscr{U}_{\theta_L,\theta_R}$ which shift the $U(1)$-charges of the states by $(-c\theta_L/3, -c\theta_R/3)$ [4, 16, 17]. Note that the sign convention used in these references differ, we choose the convention used in [16]. These spectral flow operators connect different sectors of the theory, providing explicit isomorphisms between SCFT algebras with different values for the parameter a introduced in (1). In the holomorphic sector the isomorphism is given by

$$\begin{aligned}
\mathscr{U}_{\theta_L} L_n \mathscr{U}_{\theta_L}^{-1} &= L_n + \theta_L J_n + (c/6)\theta_L^2 \delta_{n,0} \\
\mathscr{U}_{\theta_L} G_r^{\pm} \mathscr{U}_{\theta_L}^{-1} &= G_{r\pm\theta_L}^{\pm} \\
\mathscr{U}_{\theta_L} J_n \mathscr{U}_{\theta_L}^{-1} &= J_n + (c/3)\theta_L \delta_{n,0}
\end{aligned}$$

We want to point to a potential source of confusion. Let $|\phi\rangle_{(NS,NS)}$ be a state in the NSNS sector and let $|\phi\rangle_{(R,R)}$ be its image under spectral flow. The $U(1)$-charges of *both* states are defined as eigenvalues of the J_0 resp. $\widetilde{J}_0$ operator *which live in the NSNS sector*, see [18] for more details. We use the following notation:

$$\begin{aligned}
J_0|\phi\rangle_{(NS,NS)} &= q_{\phi,L}^{(NS,NS)}|\phi\rangle_{(NS,NS)}, \qquad & J_0|\phi\rangle_{(R,R)} &= q_{\phi,L}^{(R,R)}|\phi\rangle_{(R,R)} \\
\widetilde{J}_0|\phi\rangle_{(NS,NS)} &= q_{\phi,R}^{(NS,NS)}|\phi\rangle_{(NS,NS)}, \qquad & \widetilde{J}_0|\phi\rangle_{(R,R)} &= q_{\phi,R}^{(R,R)}|\phi\rangle_{(R,R)}
\end{aligned}$$

$U(1)$-charge condition	Isomorphism
$q_L^{(NS,NS)} - q_R^{(NS,NS)} \in \mathbb{Z}$	$\{\vert\phi\rangle_{(c,c)}\} \xrightarrow{\mathscr{U}_{(1/2,1/2)}} \{\vert\phi_0\rangle_{(R,R)}\}$
$q_L^{(NS,NS)} + q_R^{(NS,NS)} \in \mathbb{Z}$	$\{\vert\phi\rangle_{(a,c)}\} \xrightarrow{\mathscr{U}_{(-1/2,1/2)}} \{\vert\phi_0\rangle_{(R,R)}\}$
$q_L^{(NS,NS)} \in \mathbb{Z}/2$	$\{\vert\phi\rangle_{(c,c)}\} \xrightarrow{\mathscr{U}_{(1,0)}} \{\vert\phi\rangle_{(a,c)}\}$
$q_L^{(NS,NS)} \in \mathbb{Z}$	$\{\vert\phi\rangle_{(R,R)}\} \xrightarrow{\mathscr{U}_{(1/2,0)}} \{\vert\phi\rangle_{(NS,R)}\}$

Table 1: Isomorphisms provided by spectral flow

The spectral flow operator is well-defined, if it is a local operator with respect to all fields in the SCFT. This condition of locality translates into conditions for the $U(1)$-charges of the NSNS states [17]. Table 1 shows some of the important isomorphisms and the associated charge conditions. We use the notation $|\phi_0\rangle$ to denote the ground states of the RR sector.

The spectral flow between the RR and NSR sectors is especially important because it implements spacetime-supersymmetry. The last condition in table 1 states that the integrality of the $U(1)$-charges in the NSNS sector is a necessary condition for spacetime-supersymmetry to be present in the theory:

$$q_L^{(NS,NS)} \in \mathbb{Z} \qquad \text{(spacetime-supersymmetry condition)} \tag{9}$$

3 $\mathcal{N} = 2$ Landau-Ginzburg Theories

3.1 Chiral superfields

Before we describe Landau-Ginzburg theories, we give a short review on superspace and superfields based on chapter 12 in [19]. Consider a twodimensional field theory on flat $\mathbb{R}^2$ with time coordinate $x^0 = t$ and space coordinate $x^1 = s$. In addition to these bosonic coordinates we introduce four pairwise anticommuting coordinates

$$\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-$$

where the bar denotes complex conjugation. These six coordinates span the $(2,2)$ *superspace*. Next we define the *vector R-rotation* and the *axial*

R-rotation:

$$V \quad : \quad \theta^\pm \mapsto e^{-i\alpha}\theta^\pm, \quad \overline{\theta}^\pm \;\mapsto\; e^{i\alpha}\overline{\theta}^\pm \tag{10}$$

$$A \quad : \quad \theta^\pm \mapsto e^{\mp i\beta}\theta^\pm, \quad \overline{\theta}^\pm \;\mapsto\; e^{\pm i\beta}\overline{\theta}^\pm \tag{11}$$

Here α and β are real parameters.

Functions defined on the superspace are called *superfields*. They can be Taylor expanded in monomials in $\theta^\pm$ and $\overline{\theta}^\pm$:

$$\begin{aligned}\mathcal{F}(x^0, x^1, \theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-) &= f_0(x^0, x^1) + \theta^+ f_+(x^0, x^1) \\ &+ \theta^- f_-(x^0, x^1) + \overline{\theta}^+ f'_+(x^0, x^1) \\ &+ \overline{\theta}^- f'_-(x^0, x^1) + \theta^+\theta^- f_{+-}(x^0, x^1) + \cdots\end{aligned}$$

Due to the anticommuting property of the fermionic coordinates, there are at most 16 nonzero terms in the expansion. The action of the R-rotations on superfields is defined as follows:

$$\begin{aligned}V \quad &: \quad \mathcal{F}(x^0, x^1, \theta^\pm, \overline{\theta}^\pm) \;\mapsto\; e^{i\alpha q_V}\mathcal{F}(x^0, x^1, e^{-i\alpha}\theta^\pm, e^{i\alpha}\overline{\theta}^\pm) \\ A \quad &: \quad \mathcal{F}(x^0, x^1, \theta^\pm, \overline{\theta}^\pm) \;\mapsto\; e^{i\beta q_A}\mathcal{F}(x^0, x^1, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\overline{\theta}^\pm)\end{aligned}$$

The real numbers q_V and q_A are called *vector R-charge* resp. *axial R-charge* of $\mathcal{F}$.

Now we switch to the lightcone coordinates $x^\pm = x^0 \pm x^1$ and define

$$\partial_\pm \;=\; \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right)$$

Using these partial derivatives we define a set of four differential operators called *covariant derivatives*:

$$\begin{aligned}D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm \\ \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm\end{aligned} \tag{12}$$

Finally we define *chiral superfields* Φ and *antichiral superfields* $\overline{\Phi}$ satisfying the following equations:

$$\begin{aligned}\overline{D}_\pm\Phi = 0 &\qquad \text{(chiral superfield)} \\ D_\pm\overline{\Phi} = 0 &\qquad \text{(antichiral superfield)}\end{aligned} \tag{13}$$

3.2 Landau-Ginzburg orbifolds

The action of a $\mathcal{N}=2$ supersymmetric Landau-Ginzburg theory is defined as [19]

$$S=\int d^2zd^4\theta K(\Phi_i,\overline{\Phi}_i)+\left(\int d^2zd^2\theta W(\Phi_i)+\text{c.c.}\right) \qquad (14)$$

with $d^4\theta=d\theta^+d\theta^-d\overline{\theta}^-d\overline{\theta}^+$ and $d^2\theta=d\theta^-d\theta^+$. The $\Phi_i,\ i=1\ldots r$ are chiral superfields, $W(\Phi_i)$ is called the *superpotential.* It has the special property of quasihomogeneity, which means that $W(\Phi_i)$ satisfies the equation

$$W(\lambda^{w_i}\Phi_i)=\lambda^H W(\Phi_i)\quad \forall\lambda\in\mathbb{C},\qquad w_i,H\in\mathbb{Z} \qquad (15)$$

where the w_i are the weights of the chiral superfields. H is called the degree of the superpotential. $W(\Phi_i)$ being quasihomogeneous implies that the LG-action has a vector R-symmetry by assigning every chiral superfield R-charges as follows:

$$q_i=q(\Phi_i)=\frac{2w_i}{H} \qquad (16)$$

To see the symmetry note that $q(d^2\theta)=-2$ because of (10). Furthermore $q(W)=2$ follows from (15) and (16) with $\lambda=\exp(2\pi i/H)$.

The $\mathcal{N}=2$ LG theories are not conformally invariant, but flow into SCFT's by the renormalization group [16]. We will use the term 'LG theory' interchangeably for the high energy and the low energy theory. Thanks to $\mathcal{N}=2$ supersymmetry, the superpotential is protected from renormalization [19] and thus captures important data of the SCFT at the infrared fixed point. The (c,c)-ring of the SCFT is isomorphic to the local ring of $W(\Phi_i)$, which is the space of all monomials of Φ_i modulo setting to zero $\partial_j W(\Phi_i)$:

$$\mathcal{R}_{(c,c)}\simeq\frac{\mathbb{C}[\Phi_i]}{\langle\partial_j W(\Phi_i)\rangle} \qquad (17)$$

For the (a,a)-ring, the same relation holds with Φ_i replaced by $\overline{\Phi}_i$. On the other hand, the (a,c)-ring and the (c,a)-ring of the SCFT are trivial; this will change as soon as we introduce LG orbifolds. Due to the quasihomogeneity property, the R-charges of the LG theory flow into the left and right $U(1)$-charges of the SCFT. Since $d^2\theta$ has $U(1)$-charges (-1,-1), it follows from the neutrality of the action that $W(\Phi_i)$ has $U(1)$-charges (1,1). Then the chiral superfields must have equal left and right $U(1)$-charges at the infrared fixed point:

$$q_{i,L}=q_{i,R}=\frac{w_i}{H} \qquad (18)$$

Note that we always use subscripts L [R] to indicate that the charges are to be interpreted as left [right] charges of the low energy theory. Charges without subscript are meant to be R-charges of the high-energy theory.

The simplest example of a LG theory is characterized by the superpotential $W = \Phi^{k+2}$. The chiral ring is $\mathcal{R}_{(c,c)} = \{1, \Phi, \Phi^2, \dots, \Phi^k\}$ and we have $q_L(\Phi) = q_R(\Phi) = 1/(k+2)$. The SCFT has a central charge of $c = 3(1-q(\Phi))$ and can be identified with the level k Minimal model [20].

To make a LG theory spacetime-supersymmetric we need to project onto integral $U(1)$-charges. This can be achieved by orbifolding the theory by $\exp(2\pi i J_0)$ [21]. More precisely, let $\Gamma \cong \mathbb{Z}_H$ be the global symmetry group whose generator g acts on the superchiral fields as

$$g \ : \ \Phi_i \mapsto \omega^{w_i}\Phi_i \tag{19}$$

with $\omega = e^{2\pi i/H}$. Orbifolding is now done with respect to the generator g. The orbifolding procedure introduces sectors containing twisted states, these sectors are enumerated by a label l. In contrast to the unorbifolded LG theory there is now a nontrivial (a, c)-ring, which will play a crucial role later in this document.

To construct LG orbifolds which flow into Gepner models, we need to combine multiple Minimal models and apply the orbifolding procedure described above. This leads to LG orbifolds, where the superpotential has the form

$$W = \sum_{i=1}^{r} \Phi_i^{h_i}$$

with $h_i = H/w_i$. These models are denoted by $\mathbb{P}_{(w_1,\dots,w_r)}[H]$. Finally we need to make sure that the central charge satisfies $c = 9$. This condition is equivalent to requiring

$$H = \sum_{i=1}^{r} w_i \tag{20}$$

to hold with $r = 5$ [6]. LG orbifolds satisfying (20) are said to be of *Fermat-type* and indeed flow into Gepner models for $r = 5$. In the following we will mostly be interested in the five-variable case.

3.3 Topological twist

For $\mathcal{N} = 2$ supersymmetric theories there exists an operation called the *topological twist*, which changes the original theory into a topological field theory. We shall be brief here and refer the reader to more detailed discussions in [4, 8, 19, 22, 23]

twist signs	scalar supercharges	model name
$(-,-)$	$(G^+_{-1/2}, \widetilde{G}^+_{-1/2})$	A model
$(+,+)$	$(G^-_{-1/2}, \widetilde{G}^-_{-1/2})$	$\widetilde{A}$ model
$(+,-)$	$(G^-_{-1/2}, \widetilde{G}^+_{-1/2})$	B model
$(-,+)$	$(G^+_{-1/2}, \widetilde{G}^-_{-1/2})$	$\widetilde{B}$ model

Table 2: The four twisted models and their scalar supercharges

Conformal invariance is not necessary to perform the topological twist. Nevertheless we choose to define the twisting procedure from a SCFT point of view. For our work this is no restriction, because our theories of interest always have a conformal infrared fixed point. Twisting then means to modify the bosonic generators of the superconformal algebra as follows:

$$T(z) \mapsto T(z) \pm \frac{1}{2}\partial J(z), \qquad \widetilde{T}(z) \mapsto \widetilde{T}(z) \pm \frac{1}{2}\partial \widetilde{J}(z)$$
$$J(z) \mapsto \pm J(z), \qquad \widetilde{J}(z) \mapsto \pm \widetilde{J}(z)$$

Here it is understood that the same sign is chosen in the first and in the second line. This leaves four sign choices, which we denote with $(\pm, \pm)$. The twisting procedure modifies the conformal weights of the supercurrents. Consider the following supercharges:

$$G^{\pm}_{-1/2} = \oint G^{\pm}(z)dz, \qquad \widetilde{G}^{\pm}_{-1/2} = \oint \widetilde{G}^{\pm}(z)dz$$

One holomorphic and one antiholomorphic supercharge becomes a scalar fermionic supercharge after twisting, depending on the sign choice desribed above [23]. Table 2 shows the four different models, which we have given the names conventionally used in the literature. The $\widetilde{A}$ [$\widetilde{B}$] model is related to the A [B] model in a simple way, namely all correlators are related by an overall complex conjugation. Thus there are essentially two different topologically twisted theories: the *A model* and the *B model*. Note that the twisted supercharges are the zero modes of the twisted supercurrents due to the change in conformal weight; we have deliberately chosen to keep the mode labels unchanged in order to not cause too much confusion.

The essential step is now to interpret the scalar supercharges as BRST charges, whose cohomology gives the physical spectrum of the twisted theory [4]. It then turns out that the energy momentum tensor is BRST exact, from which it follows that all correlation functions are independent of the metric [22]. Theories with this property are called *topological field theories.* Their physical spectrum corresponds exactly to one of the four chiral rings:

$$\begin{array}{lcl} A \text{ model} & \longleftrightarrow & \mathcal{H}_{\text{phys}} = \mathcal{R}_{(c,c)} \\ \widetilde{A} \text{ model} & \longleftrightarrow & \mathcal{H}_{\text{phys}} = \mathcal{R}_{(a,a)} \\ B \text{ model} & \longleftrightarrow & \mathcal{H}_{\text{phys}} = \mathcal{R}_{(a,c)} \\ \widetilde{B} \text{ model} & \longleftrightarrow & \mathcal{H}_{\text{phys}} = \mathcal{R}_{(c,a)} \end{array} \tag{21}$$

We can interpret each of these topological theories as describing a topological sector of the full theory. Note that we have used the state-operator correspondence when interpreting the chiral rings as hilbert spaces of states.

The relation between the correlation functions in the twisted and untwisted theories has been worked out in [4]. First, there is a $U(1)$-current anomaly in the topological theory, which can be interpreted as a background charge of $(\pm c/3, \pm c/3)$. The signs again reflect the choice of the topological sector. Thus for a topological correlator to be nonzero, the total $U(1)$-charge of all insertions must compensate the background charge. From this, one can derive the following relation, which is valid if we restrict ourselves to local operator insertions:

$$\left\langle \prod_{i=1}^{n} \phi_i^{(NS,NS)} \right\rangle_{\text{twisted}} = \left\langle \phi_1^{(R,R)} \left| \prod_{i=2}^{n-1} \phi_i^{(NS,NS)} \right| \phi_n^{(R,R)} \right\rangle_{\text{untwisted}} \tag{22}$$

$|\phi_i^{(R,R)}\rangle$ are the Ramond-Ramond ground states related to the heighest weight states $|\phi_i^{(NS,NS)}\rangle$ by a spectral flow of $(\pm c/6, \pm c/6)$. We are mainly interested in the case $n = 2$, where the relation (22) can be used to relate RR-charges of D-branes and Orientifolds in the untwisted model to topological correlators in the twisted model. This also works in the high-energy regime, where we can use Landau-Ginzburg theories to study the SCFT at the infrared fixed point. Formulas for computing these RR-charges in B-type topological Landau-Ginzburg models have been developed in [11, 14]. These will be of primary importance later in this document.

3.4 Topological correlators

The charge formulas, which we will introduce later, are based on Landau-Ginzburg B-type topological correlators. For worldsheets without boundaries and crosscaps, general expressions for these correlators have been derived in [9]. These have the structure of a multidimensional residue:

$$\langle F(\Phi_i) \rangle_g = \int \frac{dx_1 dx_2 \ldots dx_r}{\partial_1 W \partial_2 W \ldots \partial_r W} F(\Phi_i) H^{g-1}(\Phi_i) \tag{23}$$

$F(\Phi_i)$ is a polynomial of the superfields, g denotes the genus of the worldsheet and $H(\Phi_i)$ is the Hessian of the superpotential $W(\Phi_i)$:

$$H(\Phi_i) = \det(\partial_j \partial_k W(\Phi_i)) \tag{24}$$

The Hessian is the unique element of maximal R-charge in the chiral ring [4]. In the following we need the dimension of the chiral ring:

$$\mu = \dim \mathcal{R}_{(c,c)} = \prod_{i=1}^{r} \left(\frac{H}{w_i} - 1 \right) = \prod_{i=1}^{r} \left(\frac{2 - q_i}{q_i} \right)$$

Let us now define the following residue:

$$\mathrm{Res}_W(F(\Phi_i)) = \langle F(\Phi_i) \rangle_{g=0}$$

It can be evaluated using the following rule: first decompose $F(\Phi_i)$ as a sum over contributions with different R-charges:

$$F(\Phi_i) = \alpha_H H(\Phi_i) + \sum_{q<q(H)} \alpha_q F_q(\Phi_i)$$

Then the residue is given by

$$\mathrm{Res}_W(F(\Phi_i)) = \alpha_H \mu \tag{25}$$

3.5 Ramond-Ramond ground states

3.5.1 Generic Landau-Ginzburg orbifolds

The formulas to calculate RR-charges of D-branes and Orientifolds, which we will introduce later, are essentially one-point functions. The single operator is an element of the (a, c)-ring related to a RR ground state in the untwisted theory. In a first step we determine all RR ground states in generic Landau-Ginzburg orbifold theories, not restricting to a particular topological sector. Following [17], we note that the states before orbifold projection have the form

$$|\phi_0\rangle^l_{(R,R)} = \prod_{lq_{i,L} \in \mathbb{Z}} \Phi_i^{n_i} |0\rangle^l_{(R,R)}$$

where $|0\rangle^l_{(R,R)}$ is the unique state with lowest $U(1)$-charges in the l-th twisted sector. Like in equation (17), the exponents n_i are constrained by the condition $\partial_j W(\Phi_i) = 0$. We see that only those chiral superfields untwisted in the l-th sector contribute to the ground states. To calculate the $U(1)$-charges of these states, we need the charges of the chiral superfields given in equation (18) and the charges of the states $|0\rangle^l_{(R,R)}$, which are calculated as follows:

$$\begin{pmatrix} J_0 \\ \widetilde{J}_0 \end{pmatrix} |0\rangle^l_{(R,R)} = \left(\pm \sum_{lq_{i,L}\notin\mathbb{Z}} \left(lq_{i,L} - [lq_{i,L}] - \frac{1}{2} \right) + \sum_{lq_{i,L}\in\mathbb{Z}} \left(q_{i,L} - \frac{1}{2} \right) \right) |0\rangle^l_{(R,R)} \tag{26}$$

Here $[x]$ denotes the greatest integer smaller than x. To find those RR ground states surviving the orbifold projection, we first perform a spectral flow into the NSNS sector, then project onto integral $U(1)$ charges and finally flow the surviving states back to the RR sector. In the special case that the central charge is a multiple of three, we can use a shortcut procedure by projecting the RR ground states directly onto $q = (2n+1)/2,\ n \in \mathbb{Z}$.

The RR ground states can be given a geometric interpretation. As we have mentioned in the introduction, there exists a correspondence between the LG orbifold theory and a CY σ-model [7], thus the RR ground states of these two theories can be related to eachother. Furthermore the RR ground states of the σ-model correspond to differential forms on the CY manifold [16] and their $U(1)$-charges relate to the hodge numbers as follows:

$$h^{p,q}(M) = \#\text{RR ground states with } (q_L, q_R) = (c/6 - p, q - c/6)$$

Table 3 shows the results of the calculation of all RR ground states for the example model $P_{(1,1,1,3,3)}[9]$ which has $c = 9$. The number of states displayed in each line contributes to a hodge number, which is given in the last column. The determination of the hodge numbers is now a simple task of adding all these contributions.

3.5.2 B-type Landau-Ginzburg orbifolds of Fermat-type

Now we restrict ourselves to the topological sector described by B-type LG orbifold theories of Fermat-type. As we have seen in (21), the physical spectrum of the B-model is given by the (a, c)-ring. The relations (7) and (8) show that the left and right $U(1)$-charges have equal modulus, but opposite sign. Since the spectral flow into the RR sector is left/right-antisymmetric, the $U(1)$-charges of the RR ground states also obey $q_L^{(R,R)} = -q_R^{(R,R)}$. Applying the orbifold projection and taking into account $c = 9$, we get the following condition:

$$q_L^{(R,R)} = -q_R^{(R,R)} \in \frac{1}{2} + \mathbb{Z} \tag{27}$$

In what follows we denote the states satisfying (27) by 'B-type RR ground states'. Now we develop some simple rules, how to determine all B-type

$W=\Phi_1^9+\Phi_2^9+\Phi_3^9+\Phi_4^3+\Phi_5^3,\quad (i,j)\in\{1,2,3\},\quad (a,b)\in\{4,5\}$				
l	$\left(q_L^{(R,R)},q_R^{(R,R)}\right)$	$\lvert\phi_0\rangle^l_{(R,R)}$	#states	$h^{p,q}(M)$
0	$(-3/2,-3/2)$	$\lvert 0\rangle^0_{(R,R)}$	1	$h^{3,0}(M)$
0	$(-1/2,-1/2)$	$\Phi_i^7\Phi_j^2\lvert 0\rangle^0_{(R,R)}$ $\Phi_i^6\Phi_j^3\lvert 0\rangle^0_{(R,R)}$ $\Phi_i^6\Phi_j^2\Phi_k\lvert 0\rangle^0_{(R,R)}$ $\dots$	112	$h^{2,1}(M)$
0	$(1/2,1/2)$	$\Phi_i^7\Phi_j^7\Phi_k^4\lvert 0\rangle^0_{(R,R)}$ $\Phi_i^7\Phi_j^7\Phi_k\Phi_a\lvert 0\rangle^0_{(R,R)}$ $\Phi_i^7\Phi_j^5\Phi_a\Phi_b\lvert 0\rangle^0_{(R,R)}$ $\dots$	112	$h^{1,2}(M)$
0	$(3/2,3/2)$	$\Phi_1^7\Phi_2^7\Phi_3^7\Phi_4\Phi_5\lvert 0\rangle^0_{(R,R)}$	1	$h^{0,3}(M)$
1	$(-3/2,3/2)$	$\lvert 0\rangle^1_{(R,R)}$	1	$h^{3,3}(M)$
2	$(-1/2,1/2)$	$\lvert 0\rangle^2_{(R,R)}$	1	$h^{2,2}(M)$
3	$(-1/2,1/2)$	$\Phi_a\lvert 0\rangle^3_{(R,R)}$	2	$h^{2,2}(M)$
4	$(-1/2,1/2)$	$\lvert 0\rangle^4_{(R,R)}$	1	$h^{2,2}(M)$
5	$(1/2,-1/2)$	$\lvert 0\rangle^5_{(R,R)}$	1	$h^{1,1}(M)$
6	$(1/2,-1/2)$	$\Phi_a\lvert 0\rangle^6_{(R,R)}$	2	$h^{1,1}(M)$
7	$(1/2,-1/2)$	$\lvert 0\rangle^7_{(R,R)}$	1	$h^{1,1}(M)$
8	$(3/2,-3/2)$	$\lvert 0\rangle^8_{(R,R)}$	1	$h^{0,0}(M)$

Table 3: RR ground states of the $P_{(1,1,1,3,3)}[9]$ model

RR ground states of any *five-variable* Fermat-type LG orbifold. Let us first restate the formula (26) as follows:

$$\begin{pmatrix} J_0 \\ \widetilde{J}_0 \end{pmatrix} |0\rangle^l_{(R,R)} = \Bigg(\underbrace{\pm \sum_{lq_{i,L}\notin\mathbb{Z}} (lq_{i,L} - [lq_{i,L}] - 1/2)}_{A} + \underbrace{\sum_{lq_{i,L}\in\mathbb{Z}} (q_{i,L} - 1/2)}_{B} \Bigg) |0\rangle^l_{(R,R)} \tag{28}$$

For $r = 5$, equation (20) is equivalent to

$$\sum_{i=1}^{5} lq_{i,L} = l \in \mathbb{Z} \tag{29}$$

To simplify the notation we introduce index sets for untwisted and twisted fields in the l-th sector:

$$I_{l,t} = \{i \in \{1,\dots,5\} : lq_{i,L} \notin \mathbb{Z}\}, \quad I_{l,u} = \{1,\dots,5\}\backslash I_{l,t} \tag{30}$$

A necessary condition for (27) to hold is $A \in 1/2 + \mathbb{Z}$. We have

$$\begin{aligned} A \in 1/2 + \mathbb{Z} \quad &\overset{(29)}{\Longleftrightarrow} \quad \exists n \in \mathbb{Z} : \sum_{i \in I_{l,u}} l q_{i,L} = (1/2)(|I_{l,t}| + 1) + n \\ &\Longleftrightarrow \quad |I_{l,t}| \in 1 + 2\mathbb{Z} \end{aligned}$$

Furthermore $|I_{l,t}| = 1$ contradicts (29) thus we get $A \in 1/2 + \mathbb{Z} \iff |I_{l,t}| \in \{3, 5\}$. Let us look at the two cases separately:

1. $|I_{l,t}| = 5 \;\Rightarrow\; B = 0$, thus $|0\rangle^l_{(R,R)}$ is the only B-type RR ground state.
2. $|I_{l,t}| = 3 \Rightarrow B \neq 0$: all B-type RR ground states are of the form $\Phi_a^{t_a}\Phi_b^{t_b}|0\rangle^l_{(R,R)}$ with $a, b \in I_{l,u}$. They need to satisfy $q_L(\Phi_a^{t_a}\Phi_b^{t_b}) = -B \Leftrightarrow (t_a+1)w_a + (t_b+1)w_b = H$. Additionally the exponents are constrained by the exactness condition: $t_{a,b} < H/w_{a,b} - 1$

To summarize, all B-type RR ground states can be found by writing down all states in each sector according to the following rules:

$$\begin{array}{lcll} |I_{l,t}| = 5 & : & |0\rangle^l_{(R,R)} & \\ |I_{l,t}| = 3 & : & \Phi_a^{t_a}\Phi_b^{t_b}|0\rangle^l_{(R,R)}, & t_{a,b} < H/w_{a,b} - 1 \\ & & & (t_a+1)w_a + (t_b+1)w_b = H \\ \text{otherwise} & : & \text{no B-type RR ground states} & \end{array} \tag{31}$$

Similar results can be found in [24]. Note that $|I_{l,t}| = n$ is equivalent to saying 'l divides exactly $(5 - n)$ of the exponent numbers h_i'. Thus the essential part of finding all B-type RR ground states has been reduced to check all sector numbers for divisors equal to the exponents in the superpotential. In the example model shown in table 3, the sectors 3 and 6 are the only ones having exactly two exponents as divisors. They are also the only sectors containing B-type RR ground states of type $\Phi_a^{t_a}\Phi_b^{t_b}|0\rangle^l_{(R,R)}$ that satisfy (27). This confirms the rule described above.

4 D-branes in B-type LG Orbifolds

4.1 Landau-Ginzburg theories with boundaries

An ordinary Landau-Ginzburg theory based on the action (14) has $\mathcal{N} = (2, 2)$ worldsheet supersymmetry. If we consider the LG theory on a Riemann surface with boundaries, the supersymmetry is expected to be broken down to at most $\mathcal{N} = 2$, because the left- and rightmoving sectors are tied together. In the following we exclusively work with boundary conditions preserving the B-type supersymmetry. In other words, the topological D-branes shall be invariant under the BRST symmetry of the B-model. It then turns out that supersymmetry is completely lost at first sight due to a boundary term, which can not be compensated by additional terms in the action containing bulk fields. The problem can be solved by introducing a fermionic superfield $\Pi(x^0, \theta^0, \overline{\theta}^0)$ living on the boundary [25]. We have chosen coordinates such that x^0, θ^0 and $\overline{\theta}^0$ span the worldsheet boundary. The covariant derivative of this superfield introduces a boundary potential $E(\Phi_i)$ by

$$D\Pi = E(\Phi_i)$$

while another boundary potential $J(\Phi_i)$ appears in the additional boundary action term:

$$S_{\partial\Sigma} = -\frac{1}{2}\int dx^0 d^2\theta \, \overline{\Pi}\, \Pi \Big|_0^\pi - \frac{i}{2}\int_{\partial\Sigma} dx^0 d\theta \, \Pi \, J(\Phi_i)_{\bar{\theta}=0}\Big|_0^\pi + \text{c.c.}$$

Supersymmetry is now restored under the condition

$$W = EJ$$

The boundary BRST operator is in close relationship to the superpotential [10]:

$$Q^2_{\text{boundary}} = W$$

The formula presented in [10] has an additional factor of i, which can be absorbed into the superpotential. In a suitable basis, $Q \equiv Q_{\text{boundary}}$ can be written as

$$Q(\Phi_i) = \begin{pmatrix} 0 & J(\Phi_i) \\ E(\Phi_i) & 0 \end{pmatrix}, \quad Q^2(\Phi_i) = W(\Phi_i) \cdot \mathbf{1} \tag{32}$$

This can be generalized such that $E(\Phi_i)$ and $J(\Phi_i)$ become matrices themselves. We conclude that all *matrix factorizations* of the superpotential describe supersymmetric D-branes.

4.2 Matrix factorizations

There exists much literature on matrix factorizations. A comprehensive review can be found in [26], for further information the reader is referred to [14, 24, 25, 27]. Please note that we now change the notation for the chiral superfields in order to be compatible to the majority of the literature on matrix factorizations:

$$x_i \equiv \Phi_i$$

As we have seen in the previous subsection, B-type D-branes (which are also called *B-branes* in the literature) are described by matrix factorizations of the superpotential. From a mathematical point of view, the operator Q acts on a free $\mathbb{Z}_2$-graded $\mathbb{C}[x_i]$-module denoted by (C, ρ). In a diagonal basis, the grading operator ρ is represented as $\rho = \mathrm{diag}\,(1, -1)$. The degree $\deg(A)$ of an operator $A : (C, \rho) \to (\widehat{C}, \widehat{\rho})$ is defined as

$$\begin{aligned} \deg(A) &= 0 \qquad \text{if} \quad \widehat{\rho} A \rho = A \\ \deg(A) &= 1 \qquad \text{if} \quad \widehat{\rho} A \rho = -A \end{aligned}$$

A is called *even* in the first and *odd* in the second case. Q is an odd endomorphism. The *rank* of a matrix factorization is defined as half the rank of Q.

B-branes in LG orbifolds need to be invariant under the orbifold action. Thus there must exist a representation of the orbifold group on the module M, with the operator γ representing the generator g, such that the following condition is satisfied:

$$\gamma Q(\omega^{w_i} x_i)\gamma^{-1} = Q(x_i), \quad \gamma^H = \mathbf{1} \tag{33}$$

See also (19) for the orbifold action on the bulk fields. Since the orbifold group $\mathbb{Z}_H$ has H representations, there exist H different D-branes corresponding to the same matrix factorization. We denote such a D-brane with (C, ρ, Q, γ).

Additionally, D-branes have to respect the vector R-symmetry. This is a necessary condition for a conformal infrared fixed point to exist [14]. Then there must exist an operator R such that

$$EQ + [R, Q] = Q, \quad E = \sum_{i=1}^{5} q_i x_i \frac{\partial}{\partial x_i} \tag{34}$$

Here q_i denotes the vector R-charge of the chiral field x_i, as defined in (16).

Let (C, ρ, Q, γ) and $(\widehat{C}, \widehat{\rho}, \widehat{Q}, \widehat{\gamma})$ be two D-branes. The states representing open strings stretching between these two D-branes are the elements of the cohomology of the $\mathbb{Z}_2$-graded complex $(\mathrm{Hom}_{\mathbb{C}[x_i]}((C, \rho), (\widehat{C}, \widehat{\rho})), \mathcal{D})$ with the odd differential $\mathcal{D}$ defined as follows:

$$\mathcal{D}\phi = \widehat{Q}\phi - (-1)^{\deg(\phi)}\phi Q \tag{35}$$

The even states are *worldsheet bosons* and the odd states are *worldsheet fermions*. Any open string state in the LG orbifold theory must be invariant under the orbifold action:

$$\widehat{\gamma}\phi(\omega^{w_i} x_i)\gamma^{-1} = \phi(x_i) \tag{36}$$

The R-charge $q(\phi)$ of an open string state ϕ is calculated using

$$E\phi + \widehat{R}\phi - \phi R = q(\phi)\phi \tag{37}$$

where R and $\widehat{R}$ are determined by equation (34).

4.3 Rank 1 factorizations

4.3.1 Tensor product branes

Let $W = x^{k+2}$ be a superpotential in one variable. The matrix factorizations

$$Q_n(x) = \begin{pmatrix} 0 & x^n \\ x^{k+2-n} & 0 \end{pmatrix}, \quad n \in \mathbb{N}_0 \tag{38}$$

are called *tensor product branes*, although they should be called Minimal branes more appropriately, since they describe D-branes in Minimal models at low energy [28]. The historical name refers to their primary use as building blocks of higher rank factorizations. We will also use them for this purpose. The integer n is called the *degree* of the matrix factorization, D-branes with $n = 1$ are conventionally called *linear tensor product branes*. The case $n = 0$ can be neglected; it corresponds to the open string vacuum, which describes the situation where there are no D-branes at all.

In the case of a one-variable superpotential, the weight of the chiral field is always $w = 1$. This is no longer true for superpotentials with more variables, therefore we need to consider general weights, so that we can later combine multiple tensor product branes. The generator of the $\mathbb{Z}_H$ orbifold group needs to satisfy (33) and is thus represented as

$$\gamma_{p,n} = \omega^p \begin{pmatrix} 1 & 0 \\ 0 & \omega^{wn} \end{pmatrix}, \quad p \in \{0, \ldots, H-1\} \tag{39}$$

where p labels the orbifold representation. In the literature it is usually denoted by m. The computation of the R-matrix according to (34) yields

$$R_n = \begin{pmatrix} \frac{1}{2} - \frac{n}{k+2} & 0 \\ 0 & -\frac{1}{2} + \frac{n}{k+2} \end{pmatrix}$$

Turning to the open string states we introduce the following notation:

$\phi_j^{T,B}$: Bosons stretching between tensor product branes

$\phi_j^{T,F}$: Fermions stretching between tensor product branes

The index j enumerates the open string states. To calculate the open strings stretching from (Q,ρ,γ) to $(\widehat{Q},\rho,\widehat{\gamma})$, we use the formula (35) and get:

$$\begin{aligned}
\phi_j^{T,B} &= \begin{pmatrix} x^{j+\frac{1}{2}(|n-\widehat{n}|-(n-\widehat{n}))} & 0 \\ 0 & x^{j+\frac{1}{2}(|n-\widehat{n}|+(n-\widehat{n}))} \end{pmatrix} \\
j &\in \{0,\ldots,\min(n,\widehat{n},h-n,h-\widehat{n})-1\} \\
\phi_j^{T,F} &= \begin{pmatrix} 0 & x^{-1-j-\frac{1}{2}(|n-\widehat{n}|-(n+\widehat{n}))} \\ -x^{h-1-j-\frac{1}{2}(|n-\widehat{n}|+(n+\widehat{n}))} & \end{pmatrix} \\
j &\in \{0,\ldots,\min(n,\widehat{n},h-n,h-\widehat{n})-1\}
\end{aligned}$$

Here we chose a different index convention compared to [28]. Note that we have assumed the grading matrix to be the same on both D-branes; flipping the sign of the grading on one brane has the single effect that the bosons become fermions and vice versa, leaving the actual matrix representation unchanged. The computation of the R-charges using (37) yields

$$q(\phi_j^{T,B}) = \frac{2}{H}\Big(wj+\frac{w}{2}(|n-\widehat{n}|)\Big) \quad (40)$$

$$q(\phi_j^{T,F}) = 1+\frac{2}{H}\Big(w(-1-j)-\frac{w}{2}(|n-\widehat{n}|)\Big) \quad (41)$$

To project the open strings onto orbifold-invariant states, we need to know the action of the orbifold generator on open strings (36):

$$\begin{aligned}
\phi_j^{T,B} &\mapsto \omega^{wj+\frac{w}{2}(|n-\widehat{n}|-(n-\widehat{n}))+\widehat{p}-p}\phi_j^{T,B} \\
\phi_j^{T,F} &\mapsto \omega^{w(-1-j)+\frac{w}{2}(-|n-\widehat{n}|-(n-\widehat{n}))+\widehat{p}-p}\phi_j^{T,F}
\end{aligned}$$

Again this is only valid if the grading matrix is equal for both D-branes. These formulas can be rewritten by inserting the R-charges (40) and (41)

$$\phi_j^T \mapsto \omega^{\frac{H}{2}\left(q(\phi_j^T)-\deg(\phi_j^T)\right)-\frac{w}{2}(n-\widehat{n})+\widehat{p}-p}\phi_j^T \quad (42)$$

yielding an expression which has the same structure for bosons and fermions.

4.3.2 Permutation branes

Let $W = x_1^{ud}+x_2^{vd}$ be a superpotential in two variables, where u and v have no common divisor: $\gcd(u,v)=1$. The matrix factorizations

$$\begin{aligned}
Q_{\mathcal{I}}(x) &= \begin{pmatrix} 0 & \prod_{j\in\mathcal{I}}(x_1^u-\eta_j x_2^v) \\ \prod_{j\in D\setminus\mathcal{I}}(x_1^u-\eta_j x_2^v) & 0 \end{pmatrix}, \quad \mathcal{I}\subset D \\
D &= \{0,\ldots,d-1\}, \quad \eta_j = e^{-i\pi(2j+1)/d}, \quad j\in\{0,\ldots,d-1\}
\end{aligned} \quad (43)$$

are called *permutation branes*. The integer $|\mathcal{I}|$ is called the *degree* of the matrix factorization, D-branes with $|\mathcal{I}| = 1$ are called *linear permutation branes*. The case $|\mathcal{I}| = 0$ describes the open string vacuum and can be neglected. D-branes with $u \neq v$ are usually called *generalized permutation branes*. A CFT description of permutation branes is currently only known for the case $u = v = 1$ with $\mathcal{I}$ containing a single set of successive integers [29]. Studying the other types of permutation branes in the LG language might be helpful to find their description in the CFT language.

Like in the case of tensor product branes, we need to consider general weights, because later we want to combine the permutation branes with other D-branes. Let $\widetilde{w} = H/d$. The generator of the $\mathbb{Z}_H$ orbifold group needs to satisfy (33) and is therefore represented as

$$\gamma_{p,\mathcal{I}} = \omega^p \begin{pmatrix} 1 & 0 \\ 0 & \omega^{\widetilde{w}|\mathcal{I}|} \end{pmatrix}, \quad p \in \{0, \dots, H-1\} \tag{44}$$

Using (34), the R-matrix is computed to be

$$R_{\mathcal{I}} = \begin{pmatrix} \frac{1}{2} - \frac{|\mathcal{I}|}{d} & 0 \\ 0 & -\frac{1}{2} + \frac{|\mathcal{I}|}{d} \end{pmatrix}$$

To calculate the open string spectrum we use the following notation:

$$\begin{aligned} \phi^{P,B}_{k_1,k_2} &: \text{ Bosons stretching between permutation branes} \\ \phi^{P,F}_{k_1,k_2} &: \text{ Fermions stretching between permutation branes} \end{aligned}$$

The indices k_1, k_2 enumerate the open string states and shall not be confused with the level numbers labeling the Minimal models. The open strings stretching from (Q, ρ, γ) to $(\widehat{Q}, \rho, \widehat{\gamma})$ are computed using (35):

$$\begin{aligned} \phi^{P,B}_{k_1,k_2} &= \begin{pmatrix} x_1^{k_1} x_2^{k_2} \prod\limits_{j \in \widehat{\mathcal{I}} \setminus (\widehat{\mathcal{I}} \cap \mathcal{I})} (x_1^u - \eta_j x_2^v) & 0 \\ 0 & x_1^{k_1} x_2^{k_2} \prod\limits_{j \in \mathcal{I} \setminus (\widehat{\mathcal{I}} \cap \mathcal{I})} (x_1^u - \eta_j x_2^v) \end{pmatrix} \\ k_1 &\in \{0, \dots, u \cdot |\widehat{\mathcal{I}} \cap \mathcal{I}| - 1\} \\ k_2 &\in \{0, \dots, v \cdot |D \setminus (\widehat{\mathcal{I}} \cup \mathcal{I})| - 1\} \\ \phi^{P,F}_{k_1,k_2} &= \begin{pmatrix} 0 & x_1^{k_1} x_2^{k_2} \prod\limits_{j \in \widehat{\mathcal{I}} \cap \mathcal{I}} (x_1^u - \eta_j x_2^v) \\ -x_1^{k_1} x_2^{k_2} \prod\limits_{j \in D \setminus (\widehat{\mathcal{I}} \cup \mathcal{I})} (x_1^u - \eta_j x_2^v) & 0 \end{pmatrix} \\ k_1 &\in \{0, \dots, u \cdot |\mathcal{I} \setminus (\widehat{\mathcal{I}} \cap \mathcal{I})| - 1\} \\ k_2 &\in \{0, \dots, v \cdot |\widehat{\mathcal{I}} \setminus (\widehat{\mathcal{I}} \cap \mathcal{I})| - 1\} \end{aligned}$$

Like for the tensor product branes, the grading matrix is assumed to be equal on both D-branes. These expressions are consistent with the results derived in [29] for the case $u = v = 1$. The evaluation of the R-charges using (37) gives

$$q(\phi^{P,B}_{k_1,k_2}) = \frac{2}{H}\Big(w_1k_1 + w_2k_2 + \frac{\widetilde{w}}{2}(|(\mathcal{I}\cup\widehat{\mathcal{I}})\backslash(\mathcal{I}\cap\widehat{\mathcal{I}})|)\Big) \tag{45}$$

$$q(\phi^{P,F}_{k_1,k_2}) = 1 + \frac{2}{H}\Big(w_1k_1 + w_2k_2 - \frac{\widetilde{w}}{2}(|(\mathcal{I}\cup\widehat{\mathcal{I}})\backslash(\mathcal{I}\cap\widehat{\mathcal{I}})|)\Big) \tag{46}$$

In the case of equal grading matrices on both D-branes, the orbifold generator acts on the open strings as follows (36):

$$\phi^{P,B}_{k_1,k_2} \mapsto \omega^{w_1k_1+w_2k_2+\frac{\widetilde{w}}{2}(|(\mathcal{I}\cup\widehat{\mathcal{I}})\backslash(\mathcal{I}\cap\widehat{\mathcal{I}})|-(|\mathcal{I}|-|\widehat{\mathcal{I}}|))+\widehat{p}-p}\phi^{P,B}_{k_1,k_2}$$

$$\phi^{P,F}_{k_1,k_2} \mapsto \omega^{w_1k_1+w_2k_2+\frac{\widetilde{w}}{2}(-|(\mathcal{I}\cup\widehat{\mathcal{I}})\backslash(\mathcal{I}\cap\widehat{\mathcal{I}})|-(|\mathcal{I}|-|\widehat{\mathcal{I}}|))+\widehat{p}-p}\phi^{P,F}_{k_1,k_2}$$

Inserting the R-charges (45) and (46) yields

$$\phi^{P}_{k_1,k_2} \mapsto \omega^{\frac{H}{2}\big(q(\phi^{P}_{k_1,k_2})-\deg(\phi^{P}_{k_1,k_2})\big)-\frac{\widetilde{w}}{2}(|\mathcal{I}|-|\widehat{\mathcal{I}}|)+\widehat{p}-p}\phi^{P}_{k_1,k_2} \tag{47}$$

4.4 The tensor product construction

Let $W = \sum_{i=1}^{r} x_i^{h_i}$ be a superpotential in r variables. The simplest way to set up matrix factorizations for this superpotential is to combine multiple rank 1 matrix factorizations. In order to do this, we decompose W into a sum of superpotentials in one or two variables, depending on the type of matrix factorization we want to use for these variables:

$$W = \sum_{i=1}^{N} W_i$$

In general $N \neq r$, because permutation branes are defined over two variables. Let $(C_i, \rho_i, Q_i, \gamma_i)$ be D-branes associated to the superpotential W_i. We use the *tensor product construction* to set up a new D-brane $(C = \otimes_{i=1}^{N} C_i, \rho, Q, \gamma)$ in the theory with the superpotential W. First, the matrix factorization and the grading have the following form:

$$Q = \sum_{i=1}^{N}\Big(\Big(\bigotimes_{j=1}^{i-1}\rho_i\Big)Q_i\Big(\bigotimes_{j=i+1}^{N}\mathbf{1}\Big)\Big) \tag{48}$$

$$\rho = \bigotimes_{j=1}^{N}\rho_i \tag{49}$$

This is the generalization of the two-variable tensor product construction described in [28]. It corresponds to tensoring together boundary states in the CFT language. In a second step we need to set up the orbifold generator γ in the tensor product theory. If we would simply define $\gamma = \otimes_{i=1}^{N}\gamma_i$, the H representations of $\mathbb{Z}_H$ would be labeled by N different numbers p_i coming from each D-brane component, and many number configurations would be equivalent. To improve the situation we note that the representation matrices have the same structure for all D-brane types discussed so far, namely

$$\gamma_i = \omega^{p_i}\widetilde{\gamma}_i$$

Then we can consistently define γ to be

$$\gamma = \omega^p \bigotimes_{i=1}^{N} \widetilde{\gamma}_i, \quad p \in \{0, \dots, H-1\} \tag{50}$$

Given the R-matrices R_i of the individual D-brane components, the R-matrix of the combined D-brane is given by

$$R = \sum_{i=1}^{N} \Big(\Big(\bigotimes_{j=1}^{i-1} \rho_i \Big) R_i \Big(\bigotimes_{j=i+1}^{N} \mathbf{1} \Big) \Big) \tag{51}$$

For the open string states, the tensor product construction works in the following way:

$$\phi = \bigodot_{i=1}^{N} \phi_i \tag{52}$$

Here $\odot$ denotes the graded tensor product defined by

$$\phi_1 \odot \phi_2 := \phi_1 \rho_1^{\deg(\phi_2)} \otimes \phi_2$$

As a consequence, the bosons are composed of an even number of fermionic components while the fermions contain an odd number of fermionic components. In particular, states with only bosonic components are always bosons, while states with only fermionic components can be either bosons or fermions, depending on the number of D-brane components. Using the decomposition of the R-matrix (51), one can confirm the additive nature of the R-charge:

$$q(\phi) = \sum_{i=1}^{N} q(\phi_i) \tag{53}$$

4.5 The phase of matrix factorizations

Now we specialize to D-branes built of tensor product branes and permutation branes using the tensor product procedure described in the previous subsection. Let us replace the orbifold label p defined in (50) by another label M:

$$M = -2p - \sum_{i\in I_T} w_i n_i - \sum_{i\in I_P} \widetilde{w}_i |\mathcal{I}_i| \qquad \text{mod } 2H \tag{54}$$

Here the index sets I_T and I_P are defined such that the first sum runs over all tensor product branes and the second over all permutation branes. M takes either even or odd values, depending on the parameters $w_i, n_i, \widetilde{w}_i$ and $\mathcal{I}_i$ defining the D-brane. With this new label, the orbifold generator γ becomes, using (39), (44) and (54),

$$\gamma = \omega^{-M/2} \bigotimes_{i\in I_T} \begin{pmatrix} \omega^{-w_i n_i/2} & 0 \\ 0 & \omega^{w_i n_i/2} \end{pmatrix} \bigotimes_{i\in I_P} \begin{pmatrix} \omega^{-\widetilde{w}_i|\mathcal{I}_i|/2} & 0 \\ 0 & \omega^{\widetilde{w}_i|\mathcal{I}_i|/2} \end{pmatrix} \tag{55}$$

Now we want to write down the orbifold action on an open string state ϕ in the tensor product theory. In the following we omit the indices enumerating the open strings. Using (42), (47), (50) and (52) we get

$$\phi \mapsto \omega^{\sum_{i\in I_T\cup I_P}\left(\frac{H}{2}(q(\phi_i)-\deg(\phi_i))\right)-\frac{1}{2}\left(\sum_{i\in I_T}(w_i n_i - \widehat{w}_i \widehat{n}_i)+\sum_{i\in I_P}\left(\widetilde{w}_i|\mathcal{I}_i| - \widehat{\widetilde{w}}_i|\widehat{\mathcal{I}}_i|\right)\right)+\widehat{p}-p}\phi$$

We rewrite this expression using (53), (54) and $\deg(\phi) = \sum_{i\in I_T\cup I_P}\deg(\phi_i)$:

$$\phi \mapsto \omega^{\frac{H}{2}(q(\phi)-\deg(\phi))+\frac{1}{2}(M-\widehat{M})}\phi \tag{56}$$

Invariant states thus have to satisfy the following relation:

$$e^{\pi i q(\phi)}(-1)^{\deg(\phi)} e^{\pi i (M-\widehat{M})/H} = 1 \tag{57}$$

When we define $\varphi = M/H$, then (57) is precisely the formula (4.35) in [14], which is derived there without reference to any particular type of D-branes. φ is called the *phase of the matrix factorization*. In subsection 5.3.3 we will see that the label M will allow us to make contact with the CFT description of D-branes.

4.6 Ramond-Ramond-charges of D-branes

D-branes are charged under the Ramond-Ramond ground states [2]. In the CFT description, these charges are realized as overlaps of boundary states with the RR ground states. The D-brane charge is thus essentially a n-tuple of complex numbers where n denotes the number of RR ground states. These n-tuples are quantized and hence form a lattice of charges [24]. If two D-branes have a charge tuple, which only differs by a sign, then they are said to be *antibranes* of eachother.

The formula to calculate the RR-charge of B-type D-branes is presented in [14]. Given a D-brane (C, ρ, Q, γ) and a RR ground state $|\phi\rangle^l$, the charge formula reads

$$\mathrm{ch}(Q,\gamma)(|\phi\rangle^l) = \frac{1}{|I_{l,u}|!}\mathrm{Res}_{W_l}\left(\phi \cdot \mathrm{Str}\left(\gamma^l(\partial Q_l)^{\wedge|I_{l,u}|}\right)\right) \tag{58}$$

Here we have $W_l(x_i) = W(x_i^u, x_i^t = 0)$ and $Q(x_i) = Q(x_i^u, x_i^t = 0)$, where x_i^u [x_i^t] are those chiral fields which are invariant [not invariant] under the orbifold action $g^l = \omega^{w_i l}$. For the definition of $I_{l,u}$ see (30). $\mathrm{Str}(\,\cdot\,) = \mathrm{Tr}_C(\rho\,\cdot\,)$ is the supertrace over the $\mathbb{Z}_2$-graded module C. ϕ is defined by $|\phi\rangle^l = \phi|0\rangle^l_{(R,R)}$. The calculation of the term $(\partial Q_l)^{\wedge|I_{l,u}|}$ is meant to be done as follows: first calculate the differential, afterwards apply the wedge product and finally dispose of the differential form $dx_1^u \wedge \cdots \wedge dx^u_{|I_{l,u}|}$, which is already present in the definition of the topological correlator (23).

A closer look at the charge formula reveals that the charge changes its sign, if we flip the grading of the D-brane and leave everything else unchanged. Thus we see that flipping the sign of the grading amounts to sending the D-brane to its antibrane.

Now we evaluate the charge formula for the special case of *five-variable* Fermat-type models and D-branes built of tensor product branes and permutation branes. According to (31) there are two types of RR ground states, to which the D-branes can couple. In the first case all fields are twisted and using (39), (44) and (50) we find

$$\mathrm{ch}(Q,\gamma)(|\phi\rangle^l) = \mathrm{Str}\,\gamma^l = \omega^{pl}\prod_{i\in I_T}(1-\omega^{w_i n_i l})\prod_{i\in I_P}(1-\omega^{\tilde{w}_i|\mathcal{I}_i|l}) \tag{59}$$

We could have chosen to use the M-label introduced in (54), but then the expression would become more complicated.

In the second case we have two untwisted fields and therefore the calculation is a bit more involved compared to the case discussed above. Looking at (31), we see that the RR ground states are of the form $|\phi\rangle^l = x_a^{t_a} x_b^{t_b}|0\rangle^l_{(R,R)}$. The first observation is that the charge vanishes if the two untwisted variables both belong to a tensor product brane:

$$|I_{l,t}| = 3 \text{ and } a, b \in I_T \quad\Rightarrow\quad \mathrm{ch}(Q,\gamma)(|\phi\rangle^l) = 0$$

In the case that a and b belong to a permutation brane, the calculation gives:

$$\begin{aligned} \mathrm{ch}(Q,\gamma)(|\phi\rangle^l) &= \omega^{pl} \prod_{i\in I_T\cap I_{l,t}} (1-\omega^{w_i n_i l}) \prod_{i\in I_P\cap I_{l,t}} (1-\omega^{\widetilde{w}_i|\mathcal{I}_i|l}) \\ &\times \frac{\widetilde{w}_a}{H} \sum_{i\in\mathcal{I}_a} \omega^{w_a(-i-1/2)(t_a+1)} \end{aligned}$$

Let us summarize these results:

$$\begin{aligned} \mathrm{ch}(Q,\gamma)(|\phi\rangle^l) = R_Q^l(t_a)\cdot\omega^{pl} \prod_{i\in I_T\cap I_{l,t}} (1-\omega^{w_i n_i l}) \prod_{i\in I_P\cap I_{l,t}} (1-\omega^{\widetilde{w}_i|\mathcal{I}_i|l}) \\ R_Q^l(t_a) = \begin{cases} 1 & \text{if } |I_{l,t}|=5 \\ 0 & \text{if } |I_{l,t}|=3 \text{ and } a,b\in I_T \\ \frac{\widetilde{w}_a}{H}\sum_{i\in\mathcal{I}_a}\omega^{w_a(-i-1/2)(t_a+1)} & \text{if } |I_{l,t}|=3 \text{ and } a,b\in \widetilde{I}_P \end{cases} \\ |\phi\rangle^l = \begin{cases} |0\rangle^l_{(R,R)} & \text{if } |I_{l,t}|=5 \\ x_a^{t_a}x_b^{t_b}|0\rangle^l_{(R,R)} & \text{if } |I_{l,t}|=3 \end{cases} \end{aligned} \tag{60}$$

Here $\widetilde{I}_P$ contains all indices, which belong to chiral fields associated to permutation branes. In the case of linear permutation branes, this result is equivalent to the expressions derived in [24].

We need to address a problem here. There can be situations, where a generalized permutation brane is defined over two variables, such that in some sectors one field is twisted and the other one is not. We want to raise the question, whether the charge formula (58) is applicable in this situation. It was rigorously proven in [14] only for the fully twisted case and it was well-motivated for the fully untwisted case. Since we consider the situation for the mixed case as unclear, we suggest not to use the charge formula for these special D-brane configurations until this issue has been cleared.

4.7 Equivalence of D-branes

Let (C,ρ,Q,γ) and $(\widehat{C},\widehat{\rho},\widehat{Q},\widehat{\gamma})$ be two D-branes. These are called *equivalent* if there exists an invertible operator $U\in GL(C,\mathbb{C}[x_1,\ldots,x_r])$ such that

$$\begin{aligned} \widehat{C} &= C \\ \widehat{\rho} &= U\rho U^{-1} \\ \widehat{Q} &= UQU^{-1} \\ \widehat{\gamma} &= U\gamma U^{-1} \end{aligned}$$

Note that $U\in GL(C,\mathbb{C}[x_1,\ldots,x_r]) \Leftrightarrow U\in GL(C,\mathbb{C})$, thus we can assume the coefficients of the U-matrix to be complex numbers. The matrix U can be decomposed into an even and an odd part:

$$U = U_B + U_F, \quad [U_B, \rho] = 0, \quad \{U_F, \rho\} = 0$$

U_B and U_F act on D-branes as follows:

$$\begin{aligned} U_B : (C, Q, \rho, \gamma) &\mapsto (C, U_B Q U_B^{-1}, \rho, U_B \gamma U_B^{-1}) = \overline{(C, U_B Q U_B^{-1}, -\rho, U_B \gamma U_B^{-1})} \\ U_F : (C, Q, \rho, \gamma) &\mapsto (C, U_F Q U_F^{-1}, -\rho, U_F \gamma U_F^{-1}) = \overline{(C, U_F Q U_F^{-1}, \rho, U_F \gamma U_F^{-1})} \end{aligned}$$

Here we have used the notation $\overline{(C, \rho, Q, \gamma)}$ to describe the antibrane of (C, ρ, Q, γ). We call two D-branes *bosonic equivalent* if they are connected by an even U-matrix and *fermionic equivalent*, if they are connected by an odd U-matrix. Additionally we will also call two matrix factorizations Q and $\widehat{Q}$ bosonic [fermionic] equivalent, if they are connected by an even [odd] U-matrix, neglecting the orbifold representation.

4.7.1 Rank 1 factorizations

Let (C, ρ, Q, γ) and $(\widehat{C}, \widehat{\rho}, \widehat{Q}, \widehat{\gamma})$ be rank 1 D-branes and let us assume that bases have been chosen such that

$$Q(x_i) = \begin{pmatrix} 0 & J(x_i) \\ E(x_i) & 0 \end{pmatrix}, \quad \widehat{Q}(x_i) = \begin{pmatrix} 0 & \widehat{J}(x_i) \\ \widehat{E}(x_i) & 0 \end{pmatrix}$$

By having a closer look at (38) and (43) we see that all the rank 1 D-branes discussed so far satisfy the following quasihomogeneity relation

$$J(\omega^{w_i} x_i) = \omega^{\widetilde{n}} J(x_i), \quad E(\omega^{w_i} x_i) = \omega^{H - \widetilde{n}} E(x_i) \tag{61}$$

where $\widetilde{n} = wn$ for tensor product branes and $\widetilde{n} = \widetilde{w}|\mathcal{I}|$ for permutation branes. Using (61) we rewrite the orbifold representation (55) such that both types of D-branes are treated in a unified manner:

$$\gamma = \omega^{-M/2} \begin{pmatrix} \omega^{-\widetilde{n}/2} & 0 \\ 0 & \omega^{\widetilde{n}/2} \end{pmatrix} \tag{62}$$

We ask now, under which conditions the two D-branes (C, ρ, Q, γ) and $(\widehat{C}, \widehat{\rho}, \widehat{Q}, \widehat{\gamma})$ are equivalent. We first analyze the bosonic and fermionic equivalence separately. In the bosonic case, U_B has the form

$$U_B = \begin{pmatrix} u_{00} & 0 \\ 0 & u_{11} \end{pmatrix}$$

and then we find

$$\widehat{Q}U_B = U_B Q \quad \Leftrightarrow \quad \begin{cases} \widehat{J}(x_i) = \alpha J(x_i) \\ \widehat{E}(x_i) = \alpha^{-1} E(x_i) \end{cases} , \quad \alpha = \frac{u_{00}}{u_{11}} \tag{63}$$

As a consequence we have $\widehat{\widetilde{n}} = \widetilde{n}$. We use this result and (62) to analyze the equation $\widehat{\gamma} U_B = U_B \gamma$ and find:

$$\widehat{\gamma} U_B = U_B \gamma \quad \Leftrightarrow \quad \widehat{M} = M \tag{64}$$

Likewise, in the fermionic case, U_F has the form

$$U_F = \begin{pmatrix} 0 & u_{01} \\ u_{10} & 0 \end{pmatrix}$$

and a similar calculation like in the bosonic case yields

$$\widehat{Q}U_F = U_F Q \quad \Leftrightarrow \quad \begin{cases} \widehat{J}(x_i) = \alpha E(x_i) \\ \widehat{E}(x_i) = \alpha^{-1} J(x_i) \end{cases} , \quad \alpha = \frac{u_{10}}{u_{01}} \tag{65}$$

In contrast to the bosonic case, we now have the relation $\widehat{\widetilde{n}} = H - \widetilde{n}$. This leads to a different condition on the M-labels:

$$\widehat{\gamma} U_F = U_F \gamma \quad \Leftrightarrow \quad \widehat{M} = M + H \tag{66}$$

A general analysis of the equation $\widehat{Q}U = UQ$ shows that two equivalent rank 1 D-branes are always bosonic or fermionic equivalent. Thus we have found the following equivalences:

$$\begin{aligned} U_B &: \quad (C, \rho, J(x_i), E(x_i), M) \simeq (C, \rho, \alpha J(x_i), \alpha^{-1} E(x_i), M) \\ U_F &: \quad (C, \rho, J(x_i), E(x_i), M) \simeq (C, -\rho, \alpha E(x_i), \alpha^{-1} J(x_i), M + H) \end{aligned}$$

4.7.2 Tensor products of rank 1 factorizations

Now we look at D-branes built of tensor product branes and permutation branes. Let (C, ρ, Q, γ) and $(\widehat{C}, \widehat{\rho}, \widehat{Q}, \widehat{\gamma})$ be such D-branes. Again we want to find conditions for these two D-branes to be equivalent. Note that the combination of two tensor product branes has rank 2, while a permutation brane has rank 1. Thus we can assume without restriction that the building blocks of both D-branes are of the same type for each variable x_i.

In the previous subsection we have found that two equivalent rank 1 D-branes are always bosonic or fermionic equivalent. Thus we make the following ansatz for the U-matrix in the tensor product theory:

$$U = \bigotimes_{i \in I_T \cup I_P} U_i, \qquad U_i \in \{U_B, U_F\} \tag{67}$$

U is even, if the number of U_F components is even and it is odd otherwise. Like in the previous subsection we define $\widetilde{n}_i = w_i n_i$ for tensor product branes and $\widetilde{n}_i = \widetilde{w}_i|\mathcal{I}_i|$ for permutation branes. The orbifold representation is then given by

$$\gamma = \omega^{-M/2} \bigotimes_{i \in I_T \cup I_P} \gamma_{M,i}, \quad \gamma_{M,i} = \begin{pmatrix} \omega^{-\widetilde{n}_i/2} & 0 \\ 0 & \omega^{\widetilde{n}_i/2} \end{pmatrix} \tag{68}$$

Now we need to distinguish two cases:

a) $J(x_i) \neq \alpha E(x_i)$ for all D-brane components

The analysis of the equation $\widehat{Q} U_B = U_B Q$ leads to a straightforward answer: the two matrix factorizations Q and $\widehat{Q}$ are bosonic [fermionic] equivalent, if and only if an even [odd] number of component pairs are fermionic equivalent and all other pairs are bosonic equivalent. We need to be more careful when analyzing the equation $\widehat{\gamma} U = U \gamma$ because the factor $\omega^{-M/2}$ in (68) is shared by all D-brane components. We have the following partial results:

$$\widehat{\gamma_M}_i U_B = U_B \gamma_{M,i} \qquad \text{: always true} \tag{69}$$

$$\widehat{\gamma_M}_i U_F = U_F \gamma_{M,i} \qquad \text{: never true} \tag{70}$$

$$(\widehat{\gamma_M}_i \otimes \widehat{\gamma_M}_j)(U_F \otimes U_F) = (U_F \otimes U_F)(\gamma_{M,i} \otimes \gamma_{M,j}) \qquad \text{: always true}$$

To see the last statement use the identity $a \otimes b = -a \otimes -b$. We still have the factors $\omega^{-M/2}$ and $\omega^{-\widehat{M}/2}$ at our disposal and we have the freedom to multiply these factors into the component equation of our choice. According to (64) and (66) we have:

$$\omega^{-\widehat{M}/2} \widehat{\gamma_M}_i \cdot U_B = U_B \cdot \omega^{-M/2} \gamma_{M,i} \quad \Leftrightarrow \quad \widehat{M} = M \tag{71}$$

$$\omega^{-\widehat{M}/2} \widehat{\gamma_M}_i \cdot U_F = U_F \cdot \omega^{-M/2} \gamma_{M,i} \quad \Leftrightarrow \quad \widehat{M} = M + H \tag{72}$$

The strategy is now as follows: if the number of fermionic equivalent component pairs is even [odd], multiply the factors $\omega^{-M/2}$ and $\omega^{-\widehat{M}/2}$ into a component equation of type (69) [(70)] and derive the condition on the M-labels from (71) [(72)]. Summarizing the results, we have found the following equivalences:

$$[U, \rho] = 0 : \quad \begin{array}{l} \widehat{J}_A(x_i) = \alpha_A E_A(x_i) \\ \widehat{E}_A(x_i) = \alpha_A^{-1} J_A(x_i) \end{array}, \quad |\{A\}| \text{ even}$$
$$\begin{array}{l} \widehat{J}_B(x_i) = \alpha_B J_B(x_i) \\ \widehat{E}_B(x_i) = \alpha_B^{-1} E_B(x_i) \end{array}, \quad |\{B\}| = |I_T \cup I_P| - |\{A\}| \tag{73}$$
$$\widehat{M} = M, \quad \widehat{\rho} = \rho$$

$$
\{U,\rho\}=0: \quad \begin{array}{l} \widehat{J}_A(x_i)=\alpha_A E_A(x_i) \\ \widehat{E}_A(x_i)=\alpha_A^{-1} J_A(x_i) \end{array}, \quad |\{A\}| \text{ odd} \\
\begin{array}{l} \widehat{J}_B(x_i)=\alpha_B J_B(x_i) \\ \widehat{E}_B(x_i)=\alpha_B^{-1} E_B(x_i) \end{array}, \quad |\{B\}|=|I_T\cup I_P|-|\{A\}| \\
\widehat{M}=M+H, \quad \widehat{\rho}=-\rho \tag{74}
$$

The condition $|\{B\}|=|I_T\cup I_P|-|\{A\}|$ just ensures that all component pairs are equivalent at all.

When we have defined D-branes as objects denoted by (C,ρ,Q,γ), we were not too precise, because we actually should define D-branes as equivalence classes of these objects. An interesting question is, if there is some natural set of representatives for these classes. Let (C,ρ,Q,γ) be an arbitrary D-brane built of tensor product branes and permutation branes. The relations (73) show that we get an equivalent D-brane, if we swap $J(x_i)$ and $E(x_i)$ in an even number of component D-branes and leave M and ρ unchanged. The relations (74) imply that swapping $J(x_i)$ and $E(x_i)$ an odd number of times and changing M to $M+H$ and ρ to $-\rho$ also yields an equivalent D-brane. Thus we can always find a representative with $\widetilde{n}_i < H/2$ for all D-brane components. In fact, let $\{[(C,\rho,Q,\gamma)]\}$ denote the set of equivalence classes, then we have the following one-to-one correspondence between these classes and a set of labels:

$$
\{[(C,\rho,Q,\gamma)]\} \;\simeq\; (\{\widetilde{n}_i : \widetilde{n}_i < H/2\}, M, \pm) \tag{75}
$$

Here the $\pm$ sign represents the choice of the grading.

b) $J(x_i)=\alpha E(x_i)$ for at least one D-brane component

In the case that a rank 1 matrix factorization is equivalent to another one with $J(x_i)=\alpha E(x_i)$, the equivalence is both bosonic and fermionic at the same time. Thus we can specialize the relations (73) and (74) as follows:

$$
[U,\rho]=0: \quad \begin{array}{l} \widehat{J}_A(x_i)=\alpha_A E_A(x_i) \\ \widehat{E}_A(x_i)=\alpha_A^{-1} J_A(x_i) \end{array}, \quad |\{A\}| \text{ arbitrary} \\
\begin{array}{l} \widehat{J}_B(x_i)=\alpha_B J_B(x_i) \\ \widehat{E}_B(x_i)=\alpha_B^{-1} E_B(x_i) \end{array}, \quad |\{B\}|=|I_T\cup I_P|+|\widetilde{I}|-|\{A\}| \\
\widetilde{I}=\{a\in I_T\cup I_P : J_a(x_i)=\alpha E_a(x_i)\} \\
\widehat{M}=M, \quad \widehat{\rho}=\rho \tag{76}
$$

$$\{U,\rho\}=0: \quad \begin{array}{l} \left.\begin{array}{l}\widehat{J}_A(x_i)=\alpha_A E_A(x_i)\\ \widehat{E}_A(x_i)=\alpha_A^{-1}J_A(x_i)\end{array}\right., \quad |\{A\}| \text{ arbitrary}\\ \left.\begin{array}{l}\widehat{J}_B(x_i)=\alpha_B J_B(x_i)\\ \widehat{E}_B(x_i)=\alpha_B^{-1}E_B(x_i)\end{array}\right., \quad |\{B\}|=|I_T\cup I_P|+|\widetilde{I}|-|\{A\}|\\ \widetilde{I}=\{a\in I_T\cup I_P : J_a(x_i)=\alpha E_a(x_i)\}\\ \widehat{M}=M+H, \quad \widehat{\rho}=-\rho \end{array} \tag{77}$$

Again the condition $|\{B\}|=|I_T\cup I_P|+|\widetilde{I}|-|\{A\}|$ ensures that all component pairs are equivalent. The introduction of the set $\widetilde{I}$ was necessary because we would get overcounting otherwise.

Let us again study the equivalence classes. If we swap $J(x_i)$ and $E(x_i)$ in an arbitrary number of component D-branes, we get equivalent D-branes in two ways: one is to leave M and ρ unchanged (76) and the other is to change M to $M+H$ and ρ to $-\rho$ (77). A first consequence is that we get a special type of equivalence, which was not present in the case a):

$$(C,\rho,Q,M) \;\simeq\; (C,-\rho,Q,M+H) \tag{78}$$

This relation says that the operation $M\mapsto M+H$ is equivalent to sending the D-brane to its antibrane. As a second consequence we see that we can always find a representative for each equivalence class, which has $\widetilde{n}_i\leq H/2$ for all D-brane components and a specific choice of grading. Therefore the set of equivalence classes is in a one-to-one correspondence with a set of labels, as follows:

$$\{[(C,\rho,Q,\gamma)]\} \;\simeq\; (\{\widetilde{n}_i:\widetilde{n}_i<H/2\},M,+) \tag{79}$$

4.8 Reducible D-branes

A D-brane (C,ρ,Q,γ) is called *reducible*, if there exists a module subspace $\widetilde{C}\subsetneq C$, $\widetilde{C}\neq\{0\}$ that is invariant under ρ, Q and γ. It is called *irreducible* if such a subspace does not exist. A reducible D-brane splits up into several components, which are sometimes called *resolved D-branes*. In terms of matrix representations, reducibility means that the matrices representing ρ, Q and γ are equivalent to matrices having the same block-diagonal structure. All rank 1 factorizations are irreducible, because Q and ρ do not commute and therefore are not simultaneously diagonalizable. The same argument does not go over to higher rank factorizations, because these matrices can be block-diagonal without actually being diagonal. In the following we will determine, under which conditions the tensor product of two rank 1 factorizations is reducible.

Let us consider the superpotential $W(x_1, x_2) = \eta_1 W_1(x_1) + \eta_2 W_2(x_2)$ with $\eta_1, \eta_2 \in \{\pm 1\}$. Furthermore let $(C_1, \rho_1, Q_1, \gamma_1)$ and $(C_2, \rho_2, Q_2, \gamma_2)$ be two rank 1 D-branes defined by

$$Q_1(x_1) = \begin{pmatrix} 0 & J_1(x_1) \\ E_1(x_1) & 0 \end{pmatrix}, \quad Q_1^2(x_1) = \eta_1 W_1(x_1) \cdot \mathbf{1}, \quad \rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$Q_2(x_2) = \begin{pmatrix} 0 & J_2(x_2) \\ E_2(x_2) & 0 \end{pmatrix}, \quad Q_2^2(x_2) = \eta_2 W_2(x_2) \cdot \mathbf{1}, \quad \rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Like in subsection 4.7.1 we assume the following quasihomogeneity relations:

$$J_a(\omega^{w_a} x_a) = \omega^{\tilde{n}_a} J_a(x_a), \quad E(\omega^{w_a} x_a) = \omega^{H - \tilde{n}_a} E_a(x_a), \quad a \in \{1, 2\}$$

We are interested in the tensor product of these two D-branes. Using (48) and (49) we find

$$Q = Q_1 \otimes \mathbf{1} + \rho_1 \otimes Q_2, \quad \rho = \rho_1 \otimes \rho_2$$

We write Q and ρ as 4×4-matrices in the canonical tensor product basis:

$$Q = \begin{pmatrix} 0 & J_2(x_2) & J_1(x_1) & 0 \\ E_2(x_2) & 0 & 0 & J_1(x_1) \\ E_1(x_1) & 0 & 0 & -J_2(x_2) \\ 0 & E_1(x_1) & -E_2(x_2) & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{80}$$

We want to know, whether this Q-matrix is equivalent to a blockdiagonal matrix factorization $\widehat{Q}$. The fermionic nature of $\widehat{Q}$ dictates its form:

$$\widehat{Q} = \begin{pmatrix} 0 & A(x_1, x_2) & 0 & 0 \\ \frac{W(x_1,x_2)}{A(x_1,x_2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & B(x_1, x_2) \\ 0 & 0 & \frac{W(x_1,x_2)}{B(x_1,x_2)} & 0 \end{pmatrix}$$

We are thus seeking a matrix U such that

$$\widehat{Q} U = U Q, \quad U \in GL(\mathbb{C}) \tag{81}$$

We restrict ourselves to bosonic U-matrices and make the following ansatz:

$$U = \begin{pmatrix} u_{00} & 0 & 0 & u_{03} \\ 0 & u_{11} & u_{12} & 0 \\ 0 & u_{21} & u_{22} & 0 \\ u_{30} & 0 & 0 & u_{33} \end{pmatrix}$$

(81) can now be analyzed as a system of 16 equations with the additional requirement that U should be a regular matrix. As a first intermediate result, we find that (81) can only be satisfied if the following condition holds

$$J_a(x_a) = \alpha_a \eta_a E_a(x_a), \quad a \in \{1,2\} \tag{82}$$

for some $\alpha_a \in \mathbb{C}$. The factor α_a can be transformed away by an equivalence transformation, thus we can assume $\alpha_a = 1$ without restriction. When we look at the definition of tensor product branes (38) and permutation branes (43) we find that (82) can only be satisfied if both D-branes are tensor product branes with $n_i = H/(2w_i)$. In this case we find after some more calculations that the U-matrix becomes

$$U = \begin{pmatrix} u_{00} & 0 & 0 & \epsilon\widetilde{\eta}u_{00} \\ 0 & u_{00} & -\epsilon\eta_2\widetilde{\eta}u_{00} & 0 \\ 0 & u_{21} & \epsilon\eta_2\widetilde{\eta}u_{21} & 0 \\ \eta_2 u_{21} & 0 & 0 & -\epsilon\eta_2\widetilde{\eta}u_{21} \end{pmatrix}, \quad \widetilde{\eta} = i^{(1+\eta_1\eta_2)/2}, \quad \epsilon \in \{\pm 1\} \tag{83}$$

Using this matrix we transform the matrix factorization and the grading given in (80):

$$\begin{aligned} \widehat{Q} = UQU^{-1} &= \begin{pmatrix} 0 & a(x_1,x_2) & 0 & 0 \\ b(x_1,x_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & a(x_1,x_2) \\ 0 & 0 & b(x_1,x_2) & 0 \end{pmatrix} \\ \widehat{\rho} = U\rho U^{-1} &= \rho \\ a(x_1,x_2) &= J_2(x_2) + \epsilon\eta_1\widetilde{\eta}J_1(x_1) \\ b(x_1,x_2) &= \eta_2(J_2(x_2) - \epsilon\eta_1\widetilde{\eta}J_1(x_1)) \end{aligned} \tag{84}$$

The conditions (82) are equivalent to $\widetilde{n}_1 = \widetilde{n}_2 = H/2$, since we can restrict ourselves to tensor product branes. Using this and (68) we can write down the orbifold representation in the canonical tensor product basis and transform it using the U-matrix:

$$\gamma = \omega^{-M/2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \widehat{\gamma} = U\gamma U^{-1} = \gamma \tag{85}$$

(84) and (85) demonstrate that the original D-brane is indeed reducible under the conditions (82). It is also true that these conditions are sufficient for a tensor product of two tensor product branes to be reducible. We should

recall that all these results have been derived under the restriction that the U-matrix is bosonic. In the case discussed above we do not expect any qualitatively new results, when more general U-matrices are taken into account, although we have not carried out the calculation explicitely. On the other hand, there are new effects coming in, when we consider tensor products of more than two rank 1 D-branes. In this situation it can happen that there are multiple ways of grouping the rank 1 D-branes into pairs, which satisfy the reducibility condition (82).

Let us make an example with the superpotential $W(x_1, x_2) = x_1^{h_1} + x_2^{h_2}$ where both exponents are even. We consider a D-brane defined by

$$\begin{aligned} Q &= \begin{pmatrix} 0 & x_1^{h_1/2} \\ x_1^{h_1/2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & x_2^{h_2/2} \\ x_2^{h_2/2} & 0 \end{pmatrix} \\ \rho &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

It satisfies the reducibility condition (82). Using (84) there exists a basis such that the D-brane is described by

$$\begin{aligned} \widehat{Q} &= \begin{pmatrix} 0 & x_2^{h_2/2} + i x_1^{h_1/2} \\ x_2^{h_2/2} - i x_1^{h_1/2} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2^{h_2/2} - i x_1^{h_1/2} \\ x_2^{h_2/2} + i x_1^{h_1/2} & 0 \end{pmatrix} \\ \widehat{\rho} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Here we have used an equivalence transformation to make the second grading matrix equal to the first one. We see that the original D-brane is equivalent to a direct sum of generalized permutation branes. This was already observed in [24].

4.9 Connection to the CFT language

Let $W = \sum_{i=1}^r x_i^{k_i+2}$ be a Landau-Ginzburg superpotential of Fermat-type and let (C, ρ, Q, γ) be a B-type D-brane which only contains tensor product branes as components. We want to make contact with the CFT description of these D-branes in the corresponding Gepner model. In [15] each D-brane is uniquely described by the symbol $B_{\boldsymbol{L},M,S,\psi}$, although some of the labels only exist when certain conditions are met. We have the following information on these four labels:

1. $\boldsymbol{L} = (L_1, \ldots, L_r)$ with $0 \leq L_i \leq k_i/2$. Let $\boldsymbol{S} = \{i : L_i = k_i/2\}$.
2. $M \in \mathbb{Z}_{2H}$ with $M = \sum_{i=1}^r w_i L_i \bmod 2$.

3. $S \in \{0, 2\}$. This label does only exist if $|\boldsymbol{S}| = 0$.

4. $\psi \in \{+, -\}$. This label does only exist if $\boldsymbol{S} \neq \emptyset$ and if $|\boldsymbol{S}|$ is even

5. A D-brane is mapped to its antibrane by $S \mapsto S + 2$, if the S-label exists, or by $M \mapsto M + H$ otherwise.

We want to compare the Landau-Ginzburg description of these D-Branes with their CFT description under the following identification of labels (see (75) and (79) for notations):

$$\begin{array}{rcll} \{i \in I_T : \widetilde{n}_i = H/2\} & \equiv & \boldsymbol{S} & \\ (\widetilde{n}_i/w_i, M, \pm) & \equiv & (L_i + 1, M, S \in \{0, 2\}) & \text{if } |\boldsymbol{S}| = 0 \\ (\widetilde{n}_i/w_i, M, +) & \equiv & (L_i + 1, M) & \text{if } |\boldsymbol{S}| > 0 \\ (\widetilde{n}_i/w_i, M, -) & \equiv & (L_i + 1, M + H) & \text{if } |\boldsymbol{S}| > 0 \end{array} \qquad (86)$$

This identification should be understood as being valid up to an exchange of the $\pm$ signs. Checking the five properties using (54), (75), (78) and (79), we see that the identification works out well. The only appararent incompatibility is that the ψ-label has not been matched in the LG description. This label is used in the CFT description to mark irreducible D-branes which are constructed by splitting special types of reducible D-branes. Towards the end of this document we will come back to this issue.

Additionally we have checked that the equivalence relations (73), (74), (76) and (77) are compatible with the corresponding identifications presented in [15]. For a single tensor product brane with $L = 0$, the label identification (86) is consistent with the one shown in [30] (up to a conventional sign). See also [29] for a LG-CFT-identification of permutation branes in unorbifolded Landau-Ginzburg theories.

5 Orientifolds in B-type LG Orbifolds

5.1 B-Parities in Landau-Ginzburg theories

Let $x^{\pm}, \theta^{\pm}, \overline{\theta}^{\pm}$ be coordinates of a $(2, 2)$ superspace, as introduced in subsection 3.1. The parity compatible with the B-type topological twist (*B-parity*) is defined as [31]

$$\Omega \;:\; (x^{\pm}, \theta^{+}, \theta^{-}, \overline{\theta}^{+}, \overline{\theta}^{-}) \;\mapsto\; (x^{\mp}, \theta^{-}, \theta^{+}, \overline{\theta}^{-}, \overline{\theta}^{+}) \qquad (87)$$

Ω is called *worldsheet parity*. It transforms the covariant derivatives, defined in (12), as $D_+ \leftrightarrow D_-$, $\overline{D}_+ \leftrightarrow \overline{D}_-$, thus we see from (13) that chiral superfields are mapped to chiral superfields.

In $\mathcal{N}=(2,2)$ Landau-Ginzburg theories, the worldsheet parity is dressed with an action on the chiral superfields denoted by τ. The resulting parity P thus has the form

$$P = \tau\Omega$$

We want P to be a symmetry of the Landau-Ginzburg action (14). In the following we assume the kinetic term to be invariant under parity (usually the Kähler potential $K(\Phi_i, \overline{\Phi_i})$ in the kinetic term is chosen, such that the target space is flat euclidean space). Let us focus on the *F-terms* containing the superpotential:

$$S_F = \int d^2z d\theta^+ d\theta^- W(\Phi_i) + \text{c.c.}$$

From (87) we see that

$$\Omega(d\theta^+ d\theta^-) = d\theta^- d\theta^+ = -d\theta^+ d\theta^-$$

and thus $\Omega^*(S_F) = -S_F$. If we want S_F to be invariant under the parity P, we have to require that τ acts holomorphically on the superchiral fields such that the superpotential transforms with a minus sign:

$$\tau^* W(\Phi_i) = W(\tau\Phi_i) = -W(\Phi_i)$$

In unorbifolded Landau-Ginzburg theories the parity has to be involutive. This condition can be relaxed in the case of Landau-Ginzburg orbifolds. In this case, parities are elements of an orbifold group extension $\widehat{\Gamma}$ [11]

$$\Gamma \to \widehat{\Gamma} \xrightarrow{\pi} \mathbb{Z}_2$$

where $\Gamma \cong \mathbb{Z}_H$ is the orbifold group. More precisely, a parity is an element $P \in \widehat{\Gamma}$ such that $\pi(P)$ is the non-trivial element of $\mathbb{Z}_2$, while a general element of the orbifold group has trivial image in $\mathbb{Z}_2$. Such a parity does only have to be involutive up to an orbifold action: $P^2 \in \Gamma$.

Let us again switch the notation for the superchiral fields: $x_i \equiv \Phi_i$. In Landau-Ginzburg orbifolds with superpotential $W = \sum_{i=1}^r x_i^h$, the most general B-parities are of the form $P_{\boldsymbol{m},\sigma,c} = \tau_{\boldsymbol{m},\sigma,c}\Omega = \widetilde{\tau}_{\boldsymbol{m},\sigma}\, g^c\, \Omega$ with

$$\begin{aligned} \widetilde{\tau}_{\boldsymbol{m},\sigma} :\ & x_i \mapsto \omega^{w_i(\frac{1}{2}+m_i)} x_{\sigma(i)}, \quad m_i + m_{\sigma(i)} = 0 \bmod \frac{H}{w_i}, \quad m_i \in \mathbb{Z} \\ \Gamma \ni g^c :\ & x_i \mapsto \omega^{w_i c} x_i, \quad c \in \{0, \dots, H-1\} \end{aligned} \tag{88}$$

where σ denotes a permutation of the chiral fields with maximal cycle length 2 [15]. See figure 1 for a visualization of the group $\widehat{\Gamma}$ containing the parities

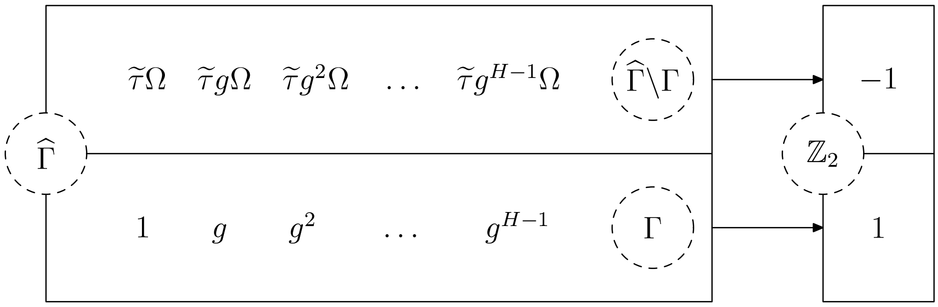

Figure 1: The orbifold group extension $\widehat{\Gamma}$

defined in (88). Orientifolds built of B-parities with $\sigma \neq \mathrm{id}$ are called *permutation orientifolds*. The weights of the chiral fields exchanged by σ need to be equal, because the parity should respect the R-symmetry:

$$w_i = w_{\sigma(i)} \tag{89}$$

5.2 Parity action on D-branes

5.2.1 Parities as functors

In section 4.2 we have seen that B-type D-branes are described by objects denoted by (C, ρ, Q, γ). The collection of these objects forms a triangulated category, the open strings are the morphisms in this category [12, 14]. There is a canonical functor in this category, the *shift functor*, which we will need in the following:

$$[1] \ : \ (C, \rho, Q, \gamma) \ \mapsto \ (C, -\rho, Q, \gamma)$$

This functor simply maps all D-branes to their antibranes.

Now we want to let the parities act on D-branes. A parity maps each D-brane of a category to some mirror D-brane, so this suggests to define the parity action on D-branes as a functor. The implementation of this functor has been worked out in [11]:

$$\mathscr{P}(\tau) \ : \ (C, \rho, Q, \gamma) \ \mapsto \ (C^*, \rho^T, -\tau^* Q^T, \chi\gamma^{-T}) \tag{90}$$

Here τ belongs to one of the H parities $P \in \widehat{\Gamma}\backslash\Gamma$. C^* denotes the dual module of C. $(\,\cdot\,)^T$ denotes the operation of *graded transposition* which is the analog of the conventional transposition for graded vector spaces. When doing actual calculations involving the graded transposition, there is a high potential for making sign errors; thus we suggest to the reader to carefully read the review on graded vector spaces presented in [11] prior to doing calculations.

Note that the rules given there for performing the graded transposition assume a specific matrix representation of the grading operator. $(\,\cdot\,)^{-T}$ denotes the composition of inversion and graded transposition, which commute for even operators such as the orbifold representation. Finally the character $\chi : \Gamma \to \mathbb{C}^x$ can be used to dress the parity action on the orbifold representation. χ corresponds to the *quantum symmetry* in Gepner models [15]. The parity functor also acts on the open strings:

$$\begin{aligned} \mathscr{P}(\tau) \;:\; \mathrm{Hom}_{\mathbb{C}[x_i]}(C_1, C_2)^\Gamma &\to \mathrm{Hom}_{\mathbb{C}[x_i]}(C_2^*, C_1^*)^\Gamma \\ \phi &\mapsto \tau^* \phi^T \end{aligned} \tag{91}$$

The superscript Γ is used here to describe Γ-invariant morphisms.

We can build a second parity functor $\mathscr{A}\mathscr{P}(\tau)$ by composing $\mathscr{P}(\tau)$ with the shift functor:

$$\mathscr{A}\mathscr{P}(\tau) = [1] \circ \mathscr{P}(\tau) \;:\; (C, \rho, Q, \gamma) \;\mapsto\; (C^*, -\rho^T, -\tau^* Q^T, \chi\gamma^{-T}) \tag{92}$$

The action on open strings remains unchanged.

We are interested to calculate the mirror images of D-branes, which are built of rank 1 factorizations. A very useful relation in this context is

$$(A_1 \otimes A_2)^T = A_1^T \otimes A_2^T$$

In the case that the parity does not exchange any chiral fields, then this relation states that the mirror image of a tensor product of rank 1 factorizations is equal to the tensor product of their mirror images. We can then restrict ourselves to study the parity action on rank 1 factorizations. In the case that the parity exchanges some chiral fields, we also need to consider the rank 2 factorizations in these variables, namely tensor products of two tensor product branes.

5.2.2 Parity action on tensor product branes

a) Orientifolds with $\sigma = \mathrm{id}$

Let us focus on a Landau-Ginzburg orbifold in one variable with superpotential $W = x^{k+2} = x^h$ and weight $w = H/h$ not restricted to be unity. In the following we will use the term 'parity' not only for the total parity P, but also for the internal part τ. According to (88) we have the following parities with $\sigma = \mathrm{id}$:

$$\tau_{m,c} \;:\; x \;\mapsto\; \omega^{w(\frac{1}{2}+m+c)} x$$

As we have seen in (90) and (92), the parity action on the orbifold representation can be dressed by a character χ. For our orbifold group $\Gamma = \mathbb{Z}_H$, it is just a phase factor: $\chi = \omega^{-M_\chi}$, $M_\chi \in \{0, \ldots, H-1\}$.

Let (C, ρ, Q, γ) be a tensor product brane. According to (38) and (39), the matrix factorization and the orbifold representation are described by

$$Q_n = \begin{pmatrix} 0 & x^n \\ x^{h-n} & 0 \end{pmatrix}, \quad \gamma_{p,n} = \omega^p \begin{pmatrix} 1 & 0 \\ 0 & \omega^{wn} \end{pmatrix}$$

The corresponding quantities of the mirror D-brane are computed to be

$$\begin{aligned} -\tau^* Q_n^T &= \begin{pmatrix} 0 & \beta x^{h-n} \\ \beta^{-1} x^n & 0 \end{pmatrix}, \quad \beta = -\omega^{-w(\frac{1}{2}+m+c)n} \\ \chi \gamma_{p,n}^{-T} &= \omega^{-M_\chi - p} \begin{pmatrix} 1 & 0 \\ 0 & \omega^{w(h-n)} \end{pmatrix} \end{aligned} \tag{93}$$

where we have used $\omega^{wh} = \omega^H = 1$. (63) shows that the factors β and β^{-1} can be transformed away by an equivalence transformation, leaving a tensor product brane in the standard representation.

b) Orientifolds with $\sigma \neq$ id

Let $W = x_1^{k_1+2} + x_2^{k_2+2} = x_1^{h_1} + x_2^{h_2}$ be a superpotential in two variables. We want to consider Orientifolds which exchange both chiral fields. Due to (89) we require $w = w_1 = w_2$. Then we have the following parities:

$$\tau_{m_1,m_2,c} \; : \; (x_1, x_2) \; \mapsto \; (\omega^{w(\frac{1}{2}+m_1+c)} x_2, \omega^{w(\frac{1}{2}+m_2+c)} x_1)$$

Let (C, ρ, Q, γ) be a tensor product of two tensor product branes. Using (38), (39), (48) and (50) the matrix factorization and the orbifold representation are given by

$$\begin{aligned} Q_{n_1,n_2} &= \begin{pmatrix} 0 & x_1^{n_1} \\ x_1^{h_1-n_1} & 0 \end{pmatrix} \otimes \mathbf{1} + \rho_1 \otimes \begin{pmatrix} 0 & x_2^{n_2} \\ x_2^{h_2-n_2} & 0 \end{pmatrix} \\ \gamma_{p,n_1,n_2} &= \omega^p \begin{pmatrix} 1 & 0 \\ 0 & \omega^{wn_1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \omega^{wn_2} \end{pmatrix} \end{aligned}$$

Acting with the parity on these quantities yields

$$\begin{aligned} -\tau^* Q_{n_1,n_2}^T &= \begin{pmatrix} 0 & \beta_1 x_2^{h_1-n_1} \\ \beta_1^{-1} x_2^{n_1} & 0 \end{pmatrix} \otimes \mathbf{1} + \rho_1 \otimes \begin{pmatrix} 0 & \beta_2 x_1^{h_2-n_2} \\ \beta_2^{-1} x_1^{n_2} & 0 \end{pmatrix} \\ \beta_i &= -\omega^{-w(\frac{1}{2}+m_i+c)n_i} \\ \chi \gamma_{p,n_1,n_2}^{-T} &= \omega^{-M_\chi - p} \begin{pmatrix} 1 & 0 \\ 0 & \omega^{-wn_1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \omega^{-wn_2} \end{pmatrix} \end{aligned} \tag{94}$$

The factors β_i can be transformed away, thus the resulting D-brane is again a tensor product of two tensor product branes in the standard representation, except that the first D-brane component contains the x_2 field, while the second contains the x_1 field.

5.2.3 Parity action on permutation branes

a) Orientifolds with $\sigma = \mathrm{id}$

Let $W = x_1^{h_1} + x_2^{h_2} = x_1^{ud} + x_2^{vd}$ be a superpotential in two variables. The parities with $\sigma = \mathrm{id}$ are the following:

$$\tau_{m_1,m_2,c} \; : \; (x_1, x_2) \; \mapsto \; (\omega^{w_1(\frac{1}{2}+m_1+c)}x_1, \omega^{w_2(\frac{1}{2}+m_2+c)}x_2)$$

Let (C, ρ, Q, γ) be a permutation brane. According to (43) and (44), the matrix factorization and the orbifold representation are given by

$$Q_{\mathcal{I}} = \begin{pmatrix} 0 & \prod_{j\in\mathcal{I}}(x_1^u - \eta_j x_2^v) \\ \prod_{j\in D\setminus\mathcal{I}}(x_1^u - \eta_j x_2^v) & 0 \end{pmatrix}, \quad \gamma_{p,\mathcal{I}} = \omega^p \begin{pmatrix} 1 & 0 \\ 0 & \omega^{\widetilde{w}|\mathcal{I}|} \end{pmatrix}$$

In the calculation of the parity action we need the following intermediate result:

$$\omega^{\widetilde{w}a}\eta_j = \omega^{\widetilde{w}a}e^{-i\pi(2j+1)/d} = \omega^{\widetilde{w}a}\omega^{\widetilde{w}(-j-\frac{1}{2})} = \omega^{\widetilde{w}(-(j-a)-\frac{1}{2})} = \eta_{j-a}$$

The matrix factorization and the orbifold representation of the mirror D-brane are now calculated to be

$$\begin{aligned} -\tau^* Q_{\mathcal{I}}^T &= \begin{pmatrix} 0 & \beta_1 \prod_{j\in D\setminus\mathcal{I}}(x_1^u - \eta_{j_*} x_2^v) \\ \beta_1^{-1}\prod_{j\in\mathcal{I}}(x_1^u - \eta_{j_*} x_2^v) & 0 \end{pmatrix} \\ j_* &= j + m_1 - m_2 \mod d \\ \beta_1 &= -\omega^{-\widetilde{w}(\frac{1}{2}+m_1+c)|\mathcal{I}|} \\ \chi\gamma_{p,\mathcal{I}}^{-T} &= \omega^{-M_\chi - p}\begin{pmatrix} 1 & 0 \\ 0 & \omega^{-\widetilde{w}|\mathcal{I}|} \end{pmatrix} \end{aligned} \tag{95}$$

The β_1 factors can be transformed away, the resulting D-brane is again a permutation brane in the standard representation. Interestingly, the m_i-labels act nontrivially on the matrix factorization. They had no effect when acting on tensor product branes.

a) Orientifolds with $\sigma \neq \mathrm{id}$

Let $W = x_1^{h_1} + x_2^{h_2}$ be a superpotential in two variables. The parities with $\sigma \neq \mathrm{id}$ are the following:

$$\tau_{m_1,m_2,c} \; : \; (x_1, x_2) \; \mapsto \; (\omega^{w(\frac{1}{2}+m_1+c)}x_2, \omega^{w(\frac{1}{2}+m_2+c)}x_1)$$

Again we require $w = w_1 = w_2$ due to (89). The matrix factorization and orbifold representation are the same as in the previous case. We need the following intermediate results:

$$\eta_j \eta_{h-1-j} = 1$$
$$\prod_{j\in\mathcal{I}}(-\eta_j)\prod_{j\in D\backslash\mathcal{I}}(-\eta_j) = \prod_{j\in D}(-\eta_j) = \prod_{j=0}^{d-1}(-\eta_j) = 1$$

Here we have defined $h = h_1 = h_2$. The parity action on the matrix factorization and the orbifold representation gives the following result:

$$\begin{aligned}
-\tau^* Q_{\mathcal{I}}^T &= \begin{pmatrix} 0 & \beta_2 \prod_{j\in D\backslash\mathcal{I}}(x_1^u - \eta_{j_*} x_2^v) \\ \beta_2^{-1}\prod_{j\in\mathcal{I}}(x_1^u - \eta_{j_*} x_2^v) & 0 \end{pmatrix} \\
j_* &= h - 1 - (j + m_1 - m_2) \mod h \\
\beta_2 &= \Big(\prod_{j\in\mathcal{I}}(-\eta_j)\Big)^{-1}(-\omega^{-\widetilde{w}(\frac{1}{2}+m_2+c)|\mathcal{I}|}) \\
\chi\gamma_{p,\mathcal{I}}^{-T} &= \omega^{-M_\chi - p}\begin{pmatrix} 1 & 0 \\ 0 & \omega^{-\widetilde{w}|\mathcal{I}|} \end{pmatrix}
\end{aligned} \tag{96}$$

The β_2 factors can be transformed away, leaving a permutation brane in the standard representation. Like in the case $\sigma = \mathrm{id}$, the m_i-labels act nontrivially on the matrix factorization.

5.3 Parity-invariant D-branes

5.3.1 The category of parity-invariant D-branes

Our final goal is to construct consistent spacetime-supersymmetric Orientifold backgrounds with D-branes. Such a background does only include D-branes which are invariant under the parity associated to the Orientifold. Therefore it is useful to define a new category, which only contains invariant D-branes. This construction was carried out in [11].

Let (C, ρ, Q, γ) be an object of the D-brane category, let $\mathcal{P}(\tau)$ be one of the two parity functors introduced in (90) and (92): $\mathcal{P}(\tau) = \mathscr{P}(\tau)$ or $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$. This D-brane is called an *invariant D-brane*, if it is equivalent to its mirror images under the parities $\mathcal{P}(\tau)$. More precisely, there exists an equivalence transformation $U(\tau)$ such that

$$U(\tau)\cdot\mathcal{P}(\tau)((\rho, Q, \gamma))\cdot U(\tau)^{-1} = (\rho, Q, \gamma) \tag{97}$$

For $\mathcal{P}(\tau) = \mathscr{P}(\tau)$ the equivalence is bosonic, for $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$ it is fermionic. In fact, we are not too precise here, because the module associated to the mirror D-brane is the dual of C, but for a strict equivalence the modules should be the same. We will nevertheless use the term 'equivalence' in this slightly generalized sense. Invariant D-branes are now described by a

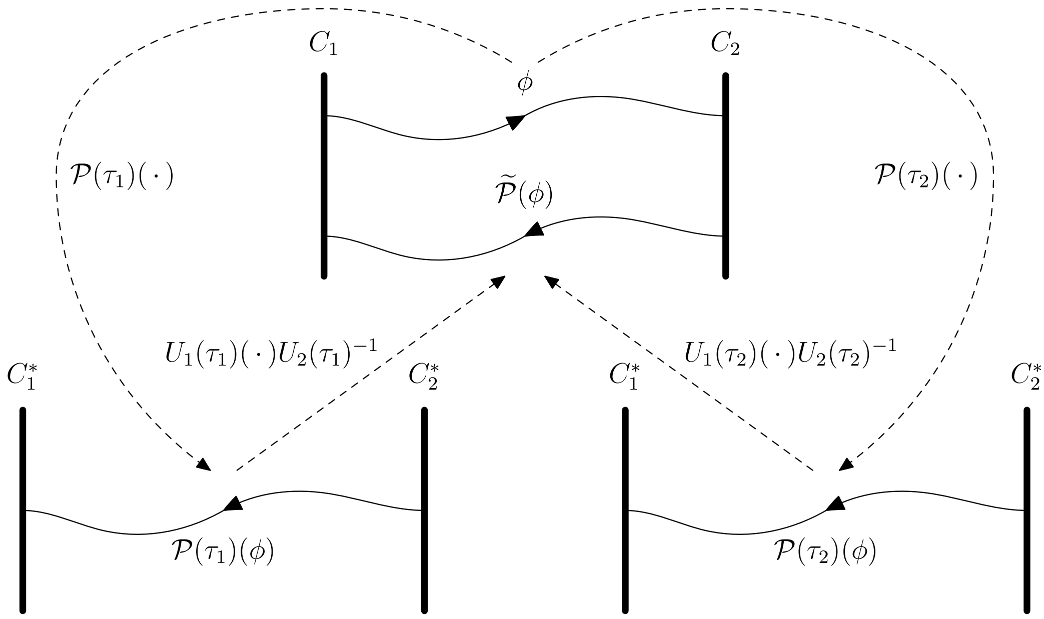

Figure 2: The parity action on open strings

quintuple $(C, \rho, Q, \gamma, U(\tau))$ and become objects of a new category, the *category of parity-invariant D-branes*. The morphisms in this new category are derived trivially from the ones in the standard D-brane category. Now we define parity functors $\widetilde{\mathcal{P}}$ in the new category:

$$\widetilde{\mathcal{P}} \; : \; (C, \rho, Q, \gamma, U) \; \mapsto \; (C, \rho, Q, \gamma, U)$$

Their action on open strings stretching between $(C_1, \rho_1, Q_1, \gamma_1, U_1(\tau))$ and $(C_2, \rho_2, Q_2, \gamma_2, U_2(\tau))$ is defined as follows:

$$\begin{aligned} \widetilde{\mathcal{P}} \; : \; \mathrm{Hom}_{\mathbb{C}[x_i]}(C_1, C_2)^{\Gamma} &\to \mathrm{Hom}_{\mathbb{C}[x_i]}(C_2, C_1)^{\Gamma} \\ \phi \; &\mapsto \; U_1(\tau) \cdot \mathcal{P}(\tau)(\phi) \cdot U_2(\tau)^{-1} \end{aligned}$$

It was shown in [11] that this map is independent of τ. See figure 2 for a graphical illustration of the parity action on open strings. $\widetilde{\mathcal{P}}$ should be an involution in the category of invariant D-branes. Thus we require the following condition to hold:

$$\widetilde{\mathcal{P}}^2(\phi) \overset{!}{=} \phi$$

As demonstrated in [11], this requirement of involutivity yields another condition, namely that

$$U(\tau)(U(\tau)^{-1})^T \iota \gamma(\tau^2)^{-1} \overset{!}{=} c(\tau) \cdot \mathrm{id}_C \tag{98}$$

holds for any invariant D-brane $(C, \rho, Q, \gamma, U(\tau))$. $\iota : C \to C^{**}$ is the canonical isomorphism, which in the standard basis has the matrix representations $\iota = \mathrm{diag}\,(1, -1)$ for $\mathcal{P}(\tau) = \mathscr{P}(\tau)$ and $\iota = \mathrm{diag}\,(1, 1)$ for $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$. $\gamma(\tau^2)$ is the operator associated to the orbifold group element τ^2. $c(\tau)$ is a phase factor, whose meaning will be explained below. Note that for fermionic U's we have $(U(\tau)^{-1})^T = -(U(\tau)^T)^{-1}$; we could have defined (98) using the opposite ordering of inversion and graded transposition, the additional sign would have been absorbed into $c(\tau)$. In [11] it was shown that the relation

$$c(\tau)^2 \chi(\tau^2) = 1 \qquad (99)$$

holds and that the function $c(\tau)$ is determined by the choice of the parity, up to a sign. Thus every category of invariant D-branes exists in two incarnations, parametrized by $c(\tau)$. We will see in subsection 5.5 that $c(\tau)$ determines the sign of the Ramond-Ramond charge of the Orientifold. Therefore by choosing one of the two categories we choose the sign of the RR-charge of the Orientifold. There is another point we should mention: equation (98) contains the U-matrices, which are D-brane-specific, so if we select one of the two categories, then one part of the D-branes satisfies (98) completely and the other part satisfies the relation only up to a sign. The latter D-branes are therefore not contained in the selected category. This selection rule will not be present anymore, as soon as we include the external spacetime dimensions. Rather we will see in subsection 5.4 that the selection rule will turn into a rule determining the gauge group of the D-brane.

5.3.2 Invariance conditions

Now we again turn to the Fermat-models and to D-branes built of tensor product branes and permutation branes. To determine whether a D-brane is invariant under the parities $\mathcal{P}(\tau)$, we need to check, whether it is equivalent to its mirror images under these parities. In section 4.7 we have analyzed, under which conditions two D-branes are equivalent to eachother, and in section 5.2 we have computed the mirror images of all D-branes we are interested in. So we have all tools at our disposal to analyze the parity invariance of these D-branes.

In a first step we want to find out, how the orbifold label M defined in (54) transforms under parities. Let us again set $\widetilde{n}_i = w_i n_i$ for tensor product branes and $\widetilde{n}_i = \widetilde{w}_i |\mathcal{I}_i|$ for permutation branes. Equation (54) then reads

$$M = -2p - \sum_{i \in I_T \cup I_P} \widetilde{n}_i \qquad \mod 2H$$

The results (93), (94), (95) and (96) show that the orbifold label p is always sent to $-M_\chi - p$. Additionally we see that $\widetilde{n}_i$ is sent to $H - \widetilde{n}_i$. Thus we have

$$\begin{aligned} \mathcal{P}(\tau)(M) &= 2(p + M_\chi) - \sum_{i \in I_T \cup I_P} (H - \widetilde{n}_i) && \mod 2H \\ \Rightarrow \quad M + \mathcal{P}(\tau)(M) &= 2M_\chi + NH && \mod 2H \end{aligned}$$

where N is again the number of D-brane components. Therefore the M-label transforms under any parity as follows:

$$M \mapsto \mathcal{P}(\tau)(M) = 2M_\chi + NH - M \qquad \mod 2H \tag{100}$$

Let (C, ρ, Q, γ) be a D-brane built of tensor product branes and permutation branes. In the following we call a D-brane *bosonic [fermionic] invariant* under a parity, if it is bosonic [fermionic] equivalent to its mirror image. Likewise we will call a matrix factorization Q bosonic [fermionic] invariant, if it is bosonic [fermonioc] equivalent to its mirror image $\mathcal{P}(\tau)(Q)$, neglecting the orbifold representation. Now we define the following numbers:

$$\begin{aligned} N_B &= |\{i \in I_T \cup I_P : Q_i \text{ is bosonic invariant}\}| \\ N_F &= |\{i \in I_T \cup I_P : Q_i \text{ is fermionic invariant}\}| \\ N_{BF} &= |\{i \in I_T \cup I_P : Q_i \text{ is bosonic and fermionic invariant}\}| \end{aligned}$$

Let us consider the two functors defined in (90) and (92) separately:

1. $\mathcal{P}(\tau) = \mathscr{P}(\tau)$:
   Here we have only bosonic invariant D-branes. (73) and (76) imply that we have $\mathscr{P}(\tau)(M) = M$ and either $N_{BF} = 0,\ N_F \in 2\mathbb{Z}$ or $N_{BF} \geq 1$. Using (100) we get the following condition on the M-label:

$$\begin{aligned} M + a2H &= 2M_\chi + NH - M \quad \forall a \in \mathbb{Z} \\ \Rightarrow \quad M &\in \left\{ M_\chi + N\frac{H}{2}, M_\chi + (N+2)\frac{H}{2} \right\} \quad \mod 2H \end{aligned}$$

2. $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$:
   Here we have only fermionic invariant D-branes. Using (74) and (77) we see that $\mathscr{A}\mathscr{P}(\tau)(M) = M + H$ and either $N_{BF} = 0,\ N_F \in 2\mathbb{Z} + 1$ or $N_{BF} \geq 1$. By the relation (100) we have

$$\begin{aligned} M + H + a2H &= 2M_\chi + NH - M \quad \forall a \in \mathbb{Z} \\ \Rightarrow \quad M &\in \left\{ M_\chi + (N+1)\frac{H}{2}, M_\chi + (N+3)\frac{H}{2} \right\} \quad \mod 2H \end{aligned}$$

Mat. fac. & Parity	bosonic invariance	fermionic invariance	bos. & ferm. invariance
TP $\sigma = \mathrm{id}$	$n = h/2$	always	$n = h/2$
$\mathrm{TP} \otimes \mathrm{TP}$ $\sigma \neq \mathrm{id}$	never	never	never
Perm $\sigma = \mathrm{id}$	$\eta_j \in \mathcal{I} \;\Leftrightarrow\; \eta_{j_*} \in D \backslash \mathcal{I}$ $j_* = j + m_1 - m_2 \quad \mathrm{mod\ d}$	$\eta_j \in \mathcal{I} \;\Leftrightarrow\; \eta_{j_*} \in \mathcal{I}$	never
Perm $\sigma \neq \mathrm{id}$	$\eta_j \in \mathcal{I} \;\Leftrightarrow\; \eta_{j_*} \in D \backslash \mathcal{I}$ $j_* = h - 1 - (j + m_1 - m_2) \quad \mathrm{mod\ h}$	$\eta_j \in \mathcal{I} \;\Leftrightarrow\; \eta_{j_*} \in \mathcal{I}$	never

Table 4: Conditions for parity invariance of rank 1 matrix factorizations

Let us summarize these results. A D-brane is parity-invariant if the following conditions hold:

$$\begin{aligned} \mathcal{P}(\tau) = \mathscr{P}(\tau): \quad & N_{BF} = 0,\ N_F \text{ even} \quad \text{or} \quad N_{BF} \geq 1 \\ & M \in \left\{ M_\chi + N\frac{H}{2}, M_\chi + (N+2)\frac{H}{2} \right\} \quad \mathrm{mod}\ 2H \\ \mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau): \quad & N_{BF} = 0,\ N_F \text{ odd} \quad \text{or} \quad N_{BF} \geq 1 \\ & M \in \left\{ M_\chi + (N+1)\frac{H}{2}, M_\chi + (N+3)\frac{H}{2} \right\} \quad \mathrm{mod}\ 2H \end{aligned} \tag{101}$$

In order to determine the numbers N_B, N_F and N_{BF} we need to find out, under which conditions the rank 1 matrix factorizations are parity-invariant. This is done by analyzing their mirror images described in (93), (94), (95), (96) and comparing these expressions with those of the original D-brane. The result of this analysis is shown in table 4.

5.3.3 Comparison with CFT results

We want to compare (101) with the CFT-based results presented in [15]. Thus we need to specialize the invariance conditions to the case of a five-variable LG orbifold theory, a parity with $\sigma = \mathrm{id}$ and D-branes built of tensor product branes only. Then we also have $N = 5$. Looking at the first row in table 4 we see that bosonic invariance implies fermionic invariance in this case. Then the condition $N_{BF} = 0,\ N_F \in 2\mathbb{Z}$ is never satisfied. Therefore $N_{BF} = 0$ implies $N_F \in 2\mathbb{Z} + 1$. Putting everything together we get

$$\begin{aligned} \mathcal{P}(\tau) = \mathscr{P}(\tau): \quad & M \in \{M_\chi \pm H/2\} \quad \mathrm{mod}\ 2H, \quad N_{BF} \geq 1 \\ \mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau): \quad & M \in \{M_\chi, M_\chi + H\} \quad \mathrm{mod}\ 2H, \end{aligned} \tag{102}$$

Comparing these conditions with the formulas (6.16) and (6.17) in [15], we find agreement up to the following incompatibilities:

- In [15] we have the condition $N_{BF} = 1$ instead of $N_{BF} \geq 1$. This can be explained by observing that D-branes are reducible if $N_{BF} > 1$ holds. The formulas (6.16) and (6.17) are only valid for irreducible D-branes.
- The condition distinguishing the two cases are not the same in both languages. In (102) we have the two functors, which select the two cases, while in [15] the distinction is made by the m_i-labels defined in (88). At first sight this seems to be a real incompatibility. On the other hand we want to emphasize that the conditions in [15] are formulated after performing a full GSO-projection, which mixes the spin-structures. In the Landau-Ginzburg approach we only make a partial GSO projection by projecting onto integral $U(1)$-charges in the NSNS sector. Additionally the spin structures are kept fixed in the Landau-Ginzburg approach [25]. Therefore it should not come as a surprise that the matching between both languages is nontrivial under these circumstances. This issue is left open for future research.

5.4 D-brane gauge groups

As we have seen in section 4.8, a reducible D-brane splits up into several irreducible ones. If the original reducible D-brane is parity-invariant, then the irreducible components do not have to be parity-invariant themselves. In this case these resolved D-branes are mapped to eachother by the parities. Thus all invariant D-branes can be built as a direct sum of two types of invariant D-branes: those which are irreducible and those which are direct sums of a non-invariant D-brane and its parity image.

Now we want to bring the external dimensions into play. We need to define an external Chan-Paton space carrying the boundary degrees of freedom associated to the external spacetime dimensions. As is well known [32], the external Chan-Paton degrees of freedom are gauge degrees of freedom. D-branes, which are direct sums of non-invariant D-branes, carry the gauge group $U(N)$, where N is the number of identical D-branes stacked together.

The situation is different for irreducible parity-invariant D-branes. In this case we expect on general grounds that the gauge group is either $O(N)$ or $SP(N/2)$ for a stack of N identical D-branes. For odd N the symplectic group is excluded. Now we would like to find out, how this case distinction shows up in the Landau-Ginzburg language. This was again developed in [11]. In section 5.3.1 we have seen that a parity functor in the category of invariant D-branes is involutive if the following condition holds for every invariant D-brane:

$$U(\tau)(U(\tau)^{-1})^T \iota\gamma(\tau^2)^{-1} \stackrel{!}{=} \epsilon c_0(\tau) \cdot \mathrm{id}_C \tag{103}$$

Here $c_0(\tau)$ is a function fixed by the chosen parity and $\epsilon \in \{\pm 1\}$ selects one of the two categories of invariant D-branes. For an invariant D-brane $(C, \rho, Q, \gamma, U_Q(\tau))$ we calculate the left hand side of (103) and we get

$$U_Q(\tau)(U_Q(\tau)^{-1})^T \iota\gamma(\tau^2)^{-1} = \epsilon_Q c_0(\tau) \cdot \mathrm{id}_C \tag{104}$$

where ϵ_Q is uniquely determined by the invariant D-brane. At first sight this yields a selection rule, because the D-brane is contained in the category parametrized by ϵ if and only if $\epsilon_Q = \epsilon$. Now let us take into account the external spacetime dimensions. We define an external Chan-Paton space $V \simeq \mathbb{C}^D$ where D denotes the number of external spacetime dimensions. This is typically set to four in order to build realistic string backgrounds. Furthermore we choose a map $U_{Q,\mathrm{ext}} : V^* \to V$ with $U_{Q,\mathrm{ext}}(U_{Q,\mathrm{ext}}^{-1})^T \in \{\pm 1\}$ and define the invariant D-brane in the full theory by

$$(\widehat{C}, \widehat{\rho}, \widehat{Q}, \widehat{\gamma}, \widehat{U}_Q(\tau)) = (V \otimes C, \mathbf{1} \otimes \rho, \mathbf{1} \otimes Q, \mathbf{1} \otimes \gamma, U_{Q,\mathrm{ext}} \otimes U_Q(\tau)) \tag{105}$$

The involutivity condition (103) in the full theory reads

$$\widehat{U}(\tau)(\widehat{U}(\tau)^{-1})^T \widehat{\iota}\widehat{\gamma}(\tau^2)^{-1} \stackrel{!}{=} \epsilon c_0(\tau) \cdot \mathrm{id}_{V \otimes C} \tag{106}$$

$\widehat{\iota} : V \otimes C \to (V \otimes C)^{**}$ is again the canonical isomorphism. The evaluation of the left hand side of (106) for the D-brane defined in (105) yields

$$\widehat{U}_Q(\tau)(\widehat{U}_Q(\tau)^{-1})^T \widehat{\iota}\widehat{\gamma}(\tau^2)^{-1} = U_{Q,\mathrm{ext}} U_{Q,\mathrm{ext}}^{-T} \epsilon_Q c_0(\tau) \cdot \mathrm{id}_{V \otimes C}$$

This D-brane is contained in the category selected by ϵ if and only if the following condition holds:

$$U_{Q,\mathrm{ext}} U_{Q,\mathrm{ext}}^{-T} = \epsilon_{Q,\mathrm{ext}} = \epsilon_Q \epsilon \tag{107}$$

The sign $\epsilon_{Q,\mathrm{ext}}$ determines the gauge group on the D-brane, which is either $O(N)$ or $SP(N/2)$ for a stack of N identical D-branes. So we have the following rule to find the gauge group of an irreducible invariant D-brane: first choose a category by fixing the sign ϵ, then calculate the sign ϵ_Q of the D-brane using (104), finally determine the sign $\epsilon_{Q,\mathrm{ext}}$ with the relation (107).

Now we calculate the sign $\epsilon_{Q,\mathrm{ext}}$ for irreducible parity-invariant tensor products of tensor product branes and permutation branes. The parity action on bulk fields has already been described in (88):

$$\tau_{\boldsymbol{m},\sigma,c} \ : \ x_i \mapsto \omega^{w_i(\frac{1}{2}+m_i+c)} x_{\sigma(i)}, \quad c \in \{0, \ldots, H-1\}$$

Therefore τ^2 acts on the bulk fields as follows:

$$\tau^2_{\boldsymbol{m},\sigma,c} \; : \; x_i \mapsto \omega^{w_i(1+2c)} x_i, \quad c \in \{0,\ldots,H-1\} \tag{108}$$

Using (68) and (108) we find that the orbifold representation $\gamma(\tau^2)$ is given by

$$\gamma(\tau^2) = \left(\omega^{-M/2} \bigotimes_{i\in I_T\cup I_P} \gamma_{M,i} \right)^{1+2c} \quad , \quad \gamma_{M,i} = \begin{pmatrix} \omega^{-\tilde{n}_i/2} & 0 \\ 0 & \omega^{\tilde{n}_i/2} \end{pmatrix} \tag{109}$$

Let $(C,\rho,Q,\gamma,U_Q(\tau))$ be an invariant rank 1 D-brane. We distinguish two cases:

1. $[U_Q(\tau),\rho] = 0$ (bosonic invariance)
   In this case we have $U_Q(\tau)(U_Q(\tau)^{-1})^T\iota = \mathrm{diag}\,(1,-1)$ and $\tilde{n} = H/2$. We now calculate ϵ_Q in (104) but we replace $\gamma(\tau^2)$ by $\gamma_M(\tau^2)$:

$$\begin{aligned} U_Q(\tau)(U_Q(\tau)^{-1})^T\iota\gamma_M(\tau^2)^{-1}/c_0(\tau) &= \begin{pmatrix} \omega^{-H/4} & 0 \\ 0 & -\omega^{H/4} \end{pmatrix}^{-(1+2c)} \\ &= (-i)^{-(1+2c)} \end{aligned} \tag{110}$$

2. $\{U_Q(\tau),\rho\} = 0$ (fermionic invariance)
   The parity action on fermionic invariant rank 1 D-branes is determined by (93), (95) and (96):

$$Q = \begin{pmatrix} 0 & J(x_i) \\ E(x_i) & 0 \end{pmatrix} \quad \mapsto \quad -\tau^* Q^T = \begin{pmatrix} 0 & \beta E(x_i) \\ \beta^{-1} J(x_i) & 0 \end{pmatrix}$$

$$\beta = \begin{cases} -\omega^{-(\frac{1}{2}+m+c)\tilde{n}} & \text{(TP brane and } \sigma = \mathrm{id}) \\ -\omega^{-(\frac{1}{2}+m_1+c)\tilde{n}} & \text{(perm. brane and } \sigma = \mathrm{id}) \\ \left(\prod_{j\in\mathcal{I}}(-\eta_j)\right)^{-1}(-\omega^{-(\frac{1}{2}+m_2+c)\tilde{n}}) & \text{(perm. brane and } \sigma \neq \mathrm{id}) \end{cases}$$

   We can simplify β in the third case by using the conditions for fermionic invariance in table 4. Consider the expression $(\prod_{j\in\mathcal{I}}(-\eta_j))$. If the common exponent h is even, then for each η_j there is a $\eta_{h-1-(j+m_1-m_2)}$ in the product and we have $|\mathcal{I}|/2$ such pairs. The calculation gives $\eta_j\eta_{h-1-(j+m_1-m_2)} = \omega^{w(m_1-m_2)}$ and then the product is evaluated to be $(\prod_{j\in\mathcal{I}}(-\eta_j)) = \omega^{(m_1-m_2)\tilde{n}/2}$. If h is odd then the product additionally contains a η_j with $j = h-1-(j+m_1-m_2)$. It is calculated to be

$\eta_j = -\omega^{w(m_1-m_2)/2}$. From this it follows that the formulas for even and odd h are the same. Now we recalculate β and we get

$$\beta = \omega^{-(\frac{1}{2}+\widetilde{m}+c)\widetilde{n}}$$

$$\widetilde{m} = \begin{cases} m & \text{(tensor product brane and } \sigma = \text{id)} \\ m_1 & \text{(permutation brane and } \sigma = \text{id)} \\ (m_1+m_2)/2 & \text{(permutation brane and } \sigma \neq \text{id)} \end{cases}$$

The $U_Q(\tau)$-matrix has the form

$$U_Q(\tau) = \begin{pmatrix} 0 & \beta u_{10} \\ u_{10} & 0 \end{pmatrix}, \quad u_{10} \in \mathbb{C}$$

From this we get the following result:

$$\begin{aligned} U_Q(\tau)(U_Q(\tau)^{-1})^T \iota\gamma_M(\tau^2)^{-1}/c_0(\tau) &= \begin{pmatrix} \beta\omega^{-\widetilde{n}/2} & 0 \\ 0 & \beta^{-1}\omega^{\widetilde{n}/2} \end{pmatrix}^{-(1+2c)} \\ &= -\omega^{\widetilde{m}\widetilde{n}} \end{aligned} \tag{111}$$

Now we want to calculate the sign $\epsilon_{Q,\text{ext}}$ of a D-brane $(C, \rho, Q, \gamma, U_Q(\tau))$ built of tensor product branes and permutation branes. From (67) we know that the U_Q-matrix is composed of bosonic and fermionic matrices $U_{Q,B,i}$ resp. $U_{Q,F,i}$. In order to calculate ϵ_Q in (104) we need to multiply the factor $(\omega^{-M/2})^{-(1+2c)}$ into one of the tensor product components. Let us define $\gamma_i(\tau^2) = (\omega^{-M/2}\gamma_{M,i})^{1+2c}$. Then we have two possibilities to get a real sign:

$$U_{Q,B,i}(\tau)(U_{Q,B,i}(\tau)^{-1})^T \iota\gamma_i(\tau^2)^{-1}/c_0(\tau) = \begin{cases} -1 & \text{if } M = H/2 \\ +1 & \text{if } M = 3H/2 \end{cases} \tag{112}$$

$$U_{Q,F,i}(\tau)(U_{Q,F,i}(\tau)^{-1})^T \iota\gamma_i(\tau^2)^{-1}/c_0(\tau) = \begin{cases} -\omega^{\widetilde{m}\widetilde{n}} & \text{if } M = 0 \\ +\omega^{\widetilde{m}\widetilde{n}} & \text{if } M = H \end{cases} \tag{113}$$

From (110) we see that each pair of $U_{Q,B,i}$ produces a minus sign. Therefore we expect a factor of the form $(-1)^{N_B/2}$ in the final expression. This expression is not correct because of two reasons: first, in the case that we have $M \in \{H/2, 3H/2\}$, one $U_{Q,B,i}$ is used to compensate the M-label factor, as shown in (112). Second, when we have $N_{BF} \geq 1$, then N_B is in general not equal to the number of $U_{Q,B,i}$ components. Let us define the following index sets:

$$\begin{aligned} \mathcal{J}_1 &= \{i \in I_T \cup I_P : Q_i \text{ is fermionic invariant}\} \\ \mathcal{J}_2 &= \{i \in \mathcal{J}_1 : Q_i \text{ is not bosonic invariant}\} \end{aligned}$$

Now we are ready to write down the result for the sign $\epsilon_{Q,\text{ext}}$

$$\begin{aligned}
\epsilon_{Q,\text{ext}} &= \epsilon\epsilon_M(-1)^{(N_B-a-b)/2}\prod_{i\in\mathcal{J}}\left(-\omega^{\widetilde{m}_i\widetilde{n}_i}\right) \\
\epsilon_M &= \begin{cases} +1 & \text{if } M\in\{0,3H/2\} \\ -1 & \text{if } M\in\{H/2,H\} \end{cases} \\
a &= \begin{cases} 0 & \text{if } M\in\{0,H\} \\ 1 & \text{if } M\in\{H/2,3H/2\} \end{cases} \\
b &= \begin{cases} 1 & \text{if } \mathcal{P}(\tau)=\mathscr{P}(\tau), \quad N_{BF}=1, \quad N_F \text{ even} \\ 1 & \text{if } \mathcal{P}(\tau)=\mathscr{A}\mathscr{P}(\tau), \quad N_{BF}=1, \quad N_F \text{ odd} \\ 0 & \text{otherwise} \end{cases} \\
\mathcal{J} &= \begin{cases} \mathcal{J}_2 & \text{if } \mathcal{P}(\tau)=\mathscr{P}(\tau), \quad N_{BF}=1, \quad N_F \text{ odd} \\ \mathcal{J}_2 & \text{if } \mathcal{P}(\tau)=\mathscr{A}\mathscr{P}(\tau), \quad N_{BF}=1, \quad N_F \text{ even} \\ \mathcal{J}_1 & \text{otherwise} \end{cases}
\end{aligned} \tag{114}$$

We have verified that this expression indeed gives a real sign in all cases. Note that these results only make sense for irreducible D-branes for which we have $N_{BF}\leq 1$. For five-variable Fermat models with $\sigma=\text{id}$ Orientifolds and D-branes built of tensor product branes only, we can compare our results with those presented in [15]. We find that the results agree up to two incompatibilities:

- There is the same problem as the one discussed in subsection 5.3.3: in the Landau-Ginzburg language we have a case distinction by the two functors, while the analog case distinction in the CFT language is based on the m_i-labels.
- In (114) we have a M-label-dependence which is not present in the CFT-based result. This problem might eventually be related to the first one.

5.5 Ramond-Ramond-charges of Orientifolds

Like D-branes, Orientifolds are charged under the Ramond-Ramond ground states [2]. The formula to calculate the RR-charges of B-type Orientifolds was developed in [11]:

$$\mathrm{ch}(\mathcal{P}(\tau))(|\phi\rangle^l)=\sum_{\tau,\tau^2=g^l}\chi(g^l)c(\tau)^{-1}\Big(\prod_{i,\tau_i^2\neq 1}(1+\tau_i)\Big)\mathrm{Res}_{W_l}(\phi\cdot C_l) \tag{115}$$

$|\phi\rangle^l$, ϕ and W_l have the same meaning as in the D-brane charge formula (58). τ_i denotes the eigenvalues of the parity matrix acting on the five chiral fields. C_l is a Crosscap operator, whose calculation will be explained below. In the following we use the term 'residue factor' for the topological correlator $\mathrm{Res}_{W_l}(\phi \cdot C_l)$.

According to the rules (31) for determining the B-type RR ground states, there are two cases we need to distinguish. In the first case all chiral fields are twisted in the l-th sector and the residue factor is trivial:

$$|I_{l_t} = 5| \quad \Rightarrow \quad \mathrm{Res}_{W_l}(\phi \cdot C_l) = 1 \tag{116}$$

In the second case we have two untwisted fields and the B-type RR ground states have the form $|\phi\rangle^l = x_a^{t_a} x_b^{t_b} |0\rangle^l_{(R,R)}$. The residue factor then contributes nontrivially to the RR-charge. Its calculation proceeds in several steps, as explained in [11]. First, the Crosscap operator vanishes if the trace of the parity matrix restricted to the untwisted fields is nontrivial:

$$\mathrm{Tr}_{W_l}\, \tau \neq 0 \quad \Rightarrow \quad C_l = 0 \tag{117}$$

In the case $\mathrm{Tr}_{W_l}\, \tau = 0$, let $x_\parallel$ and $x_\perp$ be the invariant resp. anti-invariant eigenvectors of the parity, in the subspace spanned by the untwisted fields: $\tau\, x_\parallel = x_\parallel$, $\tau\, x_\perp = -x_\perp$. Then we define a matrix factorization of the superpotential W_l as follows:

$$Q_l^P(x) = \begin{pmatrix} 0 & x_\perp \\ R(x_\perp, x_\parallel) & 0 \end{pmatrix}, \quad (Q_l^P)^2(x) = W_l(x) \cdot \mathbf{1} \tag{118}$$

The Crosscap operator is now determined by

$$C_l = \frac{1}{2} \mathrm{Str} \left(\partial Q_l^P \right)^{\wedge 2} \tag{119}$$

Like in the case of the D-brane charge formula, this expression should be understood as the coefficient of the differential form $dx_a \wedge dx_b$. Note that (119) differs from the corresponding formula in [11] by a sign. One way to check this sign is to use the index theorem presented in [11], which contains the Crosscap operator and which allows to compute the parity twisted index in the open string sector. This result can then be compared to a direct computation of the index, which does not rely on the Crosscap operator. We have done such calculations for several examples and have found agreement when using (119).

In the following we evaluate the charge formula (115) generically for all B-type orientifolds in *five-variable* Fermat-models. First, the sum in the charge formula goes over all parities squaring to some orbifold group element. Notably the models with even and odd degrees behave much differently in

H	l	Parities
odd	odd	$\tau = \widetilde{\tau}\omega^{w_i(l-1)/2}$
odd	even	$\tau = -\widetilde{\tau}\omega^{w_i(l-1)/2}$
even	odd	$\tau_1 = \widetilde{\tau}\omega^{w_i(l-1)/2},\ \tau_2 = \widetilde{\tau}\omega^{w_i(l-1+H)/2}$
even	even	-

Table 5: Parities squaring to an orbifold group element

this matter. Table 5 displays all possibilities, see (88) explaining the notation used in the table. As a consequence we get

$$\text{H even, l even} \quad \Rightarrow \quad \mathrm{ch}(\mathcal{P}(\tau))(|\phi\rangle^l) = 0 \tag{120}$$

The $\widetilde{\tau}$-part of the parity matrix in the original coordinates is block-diagonal with 1×1-blocks for each unpermuted field and with 2×2-blocks for each pair of transposed fields. In the latter case, the diagonalization leads to $\widetilde{\tau}_{1,2} = \mathrm{diag}\,(\omega^{w_i/2}, -\omega^{w_i/2})$, which shows that the associated m_i-labels for transposed fields may only show up in the residue factor. Furthermore we note that in models with odd H, the m_i-labels for unpermuted fields must vanish in order to fulfill the constraint in (88).

Using (99) we evaluate the prefactor in the charge formula:

$$\chi(g^l)c(\tau)^{-1} = \epsilon\chi(g^l)\chi(g^l)^{\frac{1}{2}} = \epsilon\omega^{\frac{-3M_\chi l}{2}}$$

ϵ is the sign which chooses one of the two categories of invariant D-branes. The M_χ label has been introduced in subsection 5.2.2.

Let us calculate the residue factor. Due to (116) we can focus on the case $|I_{l,t}| = 3$. First we need to check, how the parities in table 5 act on the untwisted fields. A short calculation then shows that all four parities behave in the exactly same way:

$$\tau:\ (x_a, x_b) \to \left(-\omega^{w_a m_a} x_{\sigma(a)}, -\omega^{w_b m_b} x_{\sigma(b)}\right)$$

Looking at (120), we see that we can restrict ourselves to the cases, where either H or l is odd, but then $h_a = 2/q_a$ is also odd. To see this in the case l odd, note that $lq_a/2 \in \mathbb{Z}$ because x_a is untwisted in the l-th sector. From this it follows that for unpermuted fields the constraint on the m_i-labels in (88) can only be satisfied with $m_a = m_b = 0$. Therefore we have $\mathrm{Tr}_{W_l}\,\tau = -2$ and then (117) implies

$$|I_{l,t}| = 3 \text{ and } a = \sigma(a) \quad \Rightarrow \quad \mathrm{Res}_{W_l}(\phi \cdot C_l) = 0 \tag{121}$$

Therefore we only need to consider the case where both untwisted fields are transposed by the parity. Then the relation (89) tells us that the weights associated to both untwisted fields must be equal. The diagonalization of the parity matrix now yields the eigenvalues $(\tau_1, \tau_2) = (1, -1)$ leading to the result $\mathrm{Tr}_{W_l}\,\tau = 0$. As we have seen, this is a necessary condition for the residue factor to be nontrivial. Due to the constraint on the m_i-labels (88) we have $\omega^{w_a m_b} = \omega^{-w_a m_a}$ and thus the parity action can be rewritten as

$$\tau:\ (x_a, x_b) \to (-\omega^{w_a m_a} x_b, -\omega^{-w_a m_a} x_a)$$

The superpotential restricted to the untwisted fields is $W_l = x_a^{h_a} + x_b^{h_a}$ and we can assume h_a to be odd, as we have explained above. Now we define the matrix factorization

$$Q_l^P(x) = \begin{pmatrix} 0 & \omega^{-\frac{w_a m_a}{2}} x_a + \omega^{\frac{w_a m_a}{2}} x_b \\ \omega^{\frac{w_a m_a}{2}} \left(\sum_{j=0}^{h_a-1} (-1)^j \omega^{w_a m_a j} x_a^{h_a-1-j} x_b^j\right) & 0 \end{pmatrix}$$

The calculation of the Crosscap operator yields

$$C_l = h_a \sum_{j=1}^{h_a-1} (-1)^j \omega^{w_a m_a j} x_a^{h_a-1-j} x_b^{j-1} \tag{122}$$

We need to evaluate the correlator $\mathrm{Res}_{W_l}(x_a^{t_a} x_b^{t_b} C_l)$. Using the second constraint in (31)

$$(t_a+1)w_a + (t_b+1)w_b = H \quad \overset{w_a=w_b}{\Rightarrow} \quad t_a + t_b = h_a - 2$$

we get $q(x_a^{t_a} x_b^{t_b}) = (h_a-2)w_a/H$. On the other hand we see from (122) that $q(C_l) = (h_a-2)w_a/H$ and therefore $q(x_a^{t_a} x_b^{t_b} C_l) = q(H_l)$ holds. Here H_l is the Hessian of the superpotential W_l, see (24) for the precise definition of the Hessian. A closer look at (122) reveals that C_l contains all possible monomials with the charge $q(H_l)$. Therefore one of the summands has to be a multiple of the Hessian and thus contributes to the residue factor. The calculation yields

$$\begin{aligned} \mathrm{Res}_{W_l}(x_a^{t_a} x_b^{t_b} C_l) &= (-1)^{t_a+1} \omega^{w_a m_a (t_a+1)} h_a \cdot \mathrm{Res}_{W_l}(x_a^{h_a-2} x_b^{h_a-2}) \\ &= \frac{w_a}{H} \omega^{w_a (h_a/2 + m_a)(t_a+1)} \end{aligned}$$

Now we are ready to write down the final result. We first introduce some index sets:

$$\begin{aligned} I_{\mathrm{id}} &= \{i \in \{1,\ldots,r\} : i = \sigma(i)\} \\ I_{\mathrm{id},E} &= \{i \in \{1,\ldots,r\} : i = \sigma(i) \text{ and } w_i \in 2\mathbb{Z}\} \\ I_{\mathrm{id},O} &= \{i \in \{1,\ldots,r\} : i = \sigma(i) \text{ and } w_i \notin 2\mathbb{Z}\} \\ I_\sigma &= \{i \in \{1,\ldots,r\} : i < \sigma(i)\} \end{aligned}$$

The Orientifold charge takes the form

$$
\begin{aligned}
\mathrm{ch}(\mathcal{P}(\tau))(|\phi\rangle^l) &= \epsilon\omega^{\frac{-3M_\chi l}{2}} R_P^l(t_a) \prod_{i\in I_\sigma\cap I_{l,t}} (1-\omega^{w_i l}) \prod_{i\in I_{\mathrm{id}}\cap I_{l,t}} (1-\omega^{(\frac{w_i+H}{2})l}) \\
&\quad \text{if H odd} \\
\mathrm{ch}(\mathcal{P}(\tau))(|\phi\rangle^l) &= \epsilon\omega^{\frac{-3M_\chi l}{2}} R_P^l(t_a) \prod_{i\in I_\sigma\cap I_{l,t}} (1-\omega^{w_i l}) \prod_{i\in I_{\mathrm{id},E}\cap I_{l,t}} (1+\omega^{w_i(m_i+\frac{l}{2})}) \\
&\quad \times \Big(\prod_{i\in I_{\mathrm{id},O}\cap I_{l,t}} (1+\omega^{w_i(m_i+\frac{l}{2})}) + \prod_{i\in I_{\mathrm{id},O}\cap I_{l,t}} (1-\omega^{w_i(m_i+\frac{l}{2})}) \Big) \\
&\quad \text{if H even and l odd} \\
\mathrm{ch}(\mathcal{P}(\tau))(|\phi\rangle^l) &= 0 \qquad \text{if H even and l even} \\
R_P^l(t_a) &= \begin{cases} 1 & \text{if } |I_{l,t}| = 5 \\ 0 & \text{if } |I_{l,t}| = 3 \text{ and } a,b\in I_{\mathrm{id}} \\ \frac{w_a}{H}\omega^{w_a(h_a/2+m_a)(t_a+1)} & \text{if } |I_{l,t}| = 3 \text{ and } a,b\in \widetilde{I}_\sigma \end{cases} \\
|\phi\rangle^l &= \begin{cases} |0\rangle^l_{(R,R)} & \text{if } |I_{l,t}| = 5 \\ x_a^{t_a} x_b^{t_b} |0\rangle^l_{(R,R)} & \text{if } |I_{l,t}| = 3 \end{cases}
\end{aligned}
\tag{123}
$$

$\widetilde{I}_\sigma$ contains the indices which belong to chiral fields exchanged by the parity. When comparing these results with the D-brane charge formula (60) we notice some similarities. First the R^l-factor looks similar in both cases, suggesting that there is some connection between the m_i-labels in permutation orientifolds and the η-values in permutation branes. On the other hand we see that the charge formulas look quite differently in the models with even H. Indeed we will see in subsection 6.2 that the only simple solutions of the tadpole constraint will be found in the models with odd H.

In some special cases we have compared the Orientifold charges with those calculated in [15]. We have found agreement for $M_\chi = 0,\ m_i = 0\ \forall i$. In all other cases the results do not match. This problem is most likely related to the issue raised in section 5.3.3, where we have noted that there exist some incompatibilities in conjunction with the m_i-labels. On the other hand, Mirror symmetry connects the quantum symmetry of a theory (parametrized by the M_χ-label) with a global symmetry of the mirror theory (parametrized by the associated m_i-labels) [15]. Thus it should not be surprising to see that we also have incompatibilities for $M_\chi \neq 0$. We suggest to restrict the usage of the charge formula to the case $M_\chi = 0,\ m_i = 0\ \forall i$, until these issues are resolved.

6 Landau-Ginzburg String Backgrounds

6.1 Spacetime-supersymmetry

To construct realistic string backgrounds we need spacetime-supersymmetry. As shown in (9), the integrality of the left and right $U(1)$-charges in the NSNS sector is a necessary condition for spacetime-supersymmetry being present in the bulk theory. A similar argument applies to the boundary sector of the theory, namely the R-charges of the open strings need to be integral

$$q(\phi) \in \mathbb{Z} \qquad \text{(spacetime-supersymmetry condition on the boundary)} \tag{124}$$

We want to study the consequences of (124). The orbifold action on open strings was shown in (56) and we need to remember that this formula is only valid if both D-branes have equal grading. Nevertheless it is straightforward to write down the orbifold action in the case that the grading of both D-branes differ:

$$\phi \mapsto \omega^{\frac{H}{2}(q(\phi)-\deg(\phi))+\frac{1}{2}(M-\widehat{M})}\phi \qquad \text{if } \widehat{\rho}=\rho \tag{125}$$

$$\phi \mapsto \omega^{\frac{H}{2}(q(\phi)-\deg(\phi)+1)+\frac{1}{2}(M-\widehat{M})}\phi \qquad \text{if } \widehat{\rho}=-\rho \tag{126}$$

To see (126) recall that flipping the grading on one D-brane has the single effect that bosons become fermions and vice versa. Now we apply the orbifold invariance condition $\phi \mapsto \phi$ to the situation of a D-brane and its image under some parity and find:

$$q(\phi) \in \mathbb{Z} \quad \Rightarrow \quad M - \widehat{M} + \alpha H \in H\mathbb{Z}, \quad \alpha \in \{0,1\}$$

Here the two values of α parametrize the choice of parity: $\mathcal{P}(\tau) = \mathscr{P}(\tau)$ resp. $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$. This is only a relative statement, the exact assignment between the α-values and the parities is not determined. From (100) we have $\widehat{M} = 2M_\chi + NH - M$ and therefore we get

$$q(\phi) \in \mathbb{Z} \quad \Rightarrow \quad M \in \frac{H}{2}\mathbb{Z} + M_\chi + \alpha\frac{H}{2} \tag{127}$$

Thus we have translated the integrality condition into a condition on the M-labels. It states that there exist at most four different M-labels which lead to a spacetime-supersymmetric background. This is consistent with the results derived in [15]. On the other hand, the CFT-based results show that given a parity, only one of the four M-labels does in fact lead to a spacetime-supersymmetric theory. Unfortunately we have not found a way to derive appropriate selection rules purely in the Landau-Ginzburg language.

Nevertheless we can make some progress by combining the knowledge of both the Landau-Ginzburg and the CFT description. Let us have a look at the spacetime-supersymmetry condition derived in the CFT-language in [15]:

$$\frac{{}_{(R,R)}\langle 0|\mathscr{B}^a\rangle_{(R,R)}}{{}_{(NS,NS)}\langle 0|\mathscr{B}^a\rangle_{(NS,NS)}} = \frac{{}_{(R,R)}\langle 0|\mathscr{C}\rangle_{(R,R)}}{{}_{(NS,NS)}\langle 0|\mathscr{C}\rangle_{(NS,NS)}} \tag{128}$$

The left hand side can be calculated in the Landau-Ginzburg approach:

$$\frac{{}_{(R,R)}\langle 0|\mathscr{B}^a\rangle_{(R,R)}}{{}_{(NS,NS)}\langle 0|\mathscr{B}^a\rangle_{(NS,NS)}} = \frac{\mathrm{ch}(Q,\gamma)(|0\rangle^1_{(R,R)})}{|\mathrm{ch}(Q,\gamma)(|0\rangle^1_{(R,R)})|}$$

Let us evaluate this expression using the charge formula (60). We can take advantage from the fact that $|I_{1,t}| = 5$ in all five-variable Fermat-models. By taking into account (54) we get the following result:

$$\frac{\mathrm{ch}(Q,\gamma)(|0\rangle^1_{(R,R)})}{|\mathrm{ch}(Q,\gamma)(|0\rangle^1_{(R,R)})|} = \exp\left(-i\frac{\pi M}{H} - iN\frac{\pi}{2}\right)$$

In the case of D-branes built of tensor product branes only, this result agrees with the one derived in [15] up to a sign in the exponent. For the right hand side of (128), the translation to the Landau-Ginzburg language is less clear because of the matching problems with the Orientifold charges, as discussed in section 5.5. Nevertheless we try the following ansatz:

$$\frac{{}_{(R,R)}\langle 0|\mathscr{C}\rangle_{(R,R)}}{{}_{(NS,NS)}\langle 0|\mathscr{C}\rangle_{(NS,NS)}} = \beta\frac{\mathrm{ch}(\mathcal{P}(\tau))(|0\rangle^1_{(R,R)})}{|\mathrm{ch}(\mathcal{P}(\tau))(|0\rangle^1_{(R,R)})|}$$

where β is some phase factor. In the following we restrict ourselves to the case $M_\chi = 0,\ m_i = 0\ \forall i$. Then the calculation of the right hand side using (123) yields

$$\frac{\mathrm{ch}(\mathcal{P}(\tau))(|0\rangle^1_{(R,R)})}{|\mathrm{ch}(\mathcal{P}(\tau))(|0\rangle^1_{(R,R)})|} = \exp\left(i\frac{\pi(2-\epsilon)}{2} - iN_\sigma\frac{\pi}{2}\right)$$

N_σ is the number of field pairs exchanged by the parity. We fix the factor β by the following idea: consider the situation with one Orientifold and one invariant D-brane which cancel the tadpoles with $\mathrm{ch}(Q,\gamma) = -\mathrm{ch}(\mathcal{P}(\tau))$. Then we can assume the background to be spacetime-supersymmetric. In this case β has to be set to -1. The supersymmetry condition (128) now reads

$$-\frac{\pi M}{H} - N\frac{\pi}{2} = -\frac{\pi(2-\epsilon)}{2} - N_\sigma\frac{\pi}{2} \mod 2\pi$$

This result is not compatible with (127), which states that the exchange of the parities $\mathscr{P}(\tau)$ and $\mathscr{A}\mathscr{P}(\tau)$ should be accompanied by shifting the M-label by $H/2$ in order to keep spacetime-supersymmetry. We can improve the situation by assigning to both parities two β-values, which differ by a phase of $\pi/2$. Thus we propose that the spacetime-supersymmetry condition should have the following form:

$$-\frac{\pi M}{H} - N\frac{\pi}{2} = \frac{\pi(2-\epsilon \pm 1)}{2} - N_\sigma \frac{\pi}{2} \mod 2\pi \quad \text{if } \mathcal{P}(\tau) = \mathscr{P}(\tau) \qquad (129)$$

$$-\frac{\pi M}{H} - N\frac{\pi}{2} = -\frac{\pi(2-\epsilon)}{2} - N_\sigma \frac{\pi}{2} \mod 2\pi \quad \text{if } \mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau) \qquad (130)$$

$$M_\chi = 0, \; m_i = 0 \; \forall i$$

For $N = 5$ and $N_\sigma = 0$ we can compare (130) with the results presented in [15] and find agreement. The sign ambiguity in (129) can not be fixed at the moment. How should the additional phase be interpreted? The possibility of absorbing the phase into the Orientifold charge is ruled out, because the reality of the parity twisted index would be spoiled. This suggests that we have the following relation in the case $\mathcal{P}(\tau) = \mathscr{P}(\tau)$:

$${}_{(NS,NS)}\langle 0|\mathscr{C}\rangle_{(NS,NS)} = \pm i|\mathrm{ch}(\mathcal{P}(\tau))(|0\rangle^1_{(R,R)})|$$

From a physical point of view, the left hand side of this equation is interpreted as a tension of the Orientifold plane which should be real. On the other hand, the right hand side is purely imaginary. Notably, similar effects have been observed in [15] where additional phase factors had to be introduced to make the Orientifold tension real. We suggest to prefer using the parity $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$ for constructing string backgrounds, until the validity of the additional phase has been verified.

6.2 Simple tadpolefree backgrounds

In the following we restrict ourselves to *five-variable* Fermat-models. We are looking for configurations with Orientifolds and D-branes that solve the tadpole constraint:

$$\frac{1}{D}\sum_i \mathrm{ch}(Q_i, \gamma_i) + \mathrm{ch}(\mathcal{P}(\tau)) = 0 \qquad (131)$$

D denotes the number of external dimensions and the charges are meant to be calculated in the internal theory. The factor of $1/D$ takes the external dimensions into account [15]. First we want to concentrate on backgrounds

Model	4 D-branes, $p=0$	Orientifold, $\mathcal{P}(\tau)=\mathscr{A}\mathscr{P}(\tau)$, $\epsilon=-1,\ M_\chi=0,\ m_i=0\ \forall i$
$\mathbb{P}_{(1,1,1,1,1)}[5]$	$(3,3,3,3,3)$	$\sigma=\mathrm{id}$
	$(3,\{\eta_2\},\{\eta_2\})$	$\sigma=(2\leftrightarrow 3, 4\leftrightarrow 5)$
$\mathbb{P}_{(1,1,1,3,3)}[9]$	$(5,5,5,6,6)$	$\sigma=\mathrm{id}$
	$(5,\{\eta_4\},\{\eta_1\})$	$\sigma=(2\leftrightarrow 3, 4\leftrightarrow 5)$
$\mathbb{P}_{(1,1,3,5,5)}[15]$	$(8,8,9,10,10)$	$\sigma=\mathrm{id}$
	$(\{\eta_7\},9,\{\eta_1\})$	$\sigma=(1\leftrightarrow 2, 4\leftrightarrow 5)$
$\mathbb{P}_{(1,3,3,3,5)}[15]$	$(8,9,9,9,10)$	$\sigma=\mathrm{id}$
$\mathbb{P}_{(1,3,3,7,7)}[21]$	$(11,12,12,14,14)$	$\sigma=\mathrm{id}$
	$(11,\{\eta_3\},\{\eta_1\})$	$\sigma=(2\leftrightarrow 3, 4\leftrightarrow 5)$
$\mathbb{P}_{(1,5,9,15,15)}[45]$	$(23,25,27,30,30)$	$\sigma=\mathrm{id}$

Table 6: Simple solutions of the tadpole constraint, including non-invariant configurations

consisting of an Orientifold and one stack of identical irreducible invariant D-branes. We call these configurations 'simple backgrounds'. In models with H even the charge formulas (60) and (123) differ too much to allow simple solutions. Therefore we can restrict ourselves to the six models with H odd: $\mathbb{P}_{(1,1,1,1,1)}[5]$, $\mathbb{P}_{(1,1,1,3,3)}[9]$, $\mathbb{P}_{(1,1,3,5,5)}[15]$, $\mathbb{P}_{(1,3,3,3,5)}[15]$, $\mathbb{P}_{(1,3,3,7,7)}[21]$ and $\mathbb{P}_{(1,5,9,15,15)}[45]$.

From (101) we see that in these models there are no irreducible D-branes invariant under the parity $\mathcal{P}(\tau)=\mathscr{P}(\tau)$. Additionally D-branes in these models can not contain components having middle degree $\widetilde{n}_i=H/2$; thus none of the components can be bosonic invariant according to table 4. Equivalently we have $N_{BF}=N_B=0$. Then using (101) we deduce that irreducible invariant D-branes do only exist if the number of components N is odd. In five-variable Fermat-models there are only two possibilities: either we have zero or two permutation branes.

By having another look at the charge formulas we notice that each factor in the D-brane charge formula associated to a permutation brane should be matched by a factor in the Orientifold charge formula associated to the exchange of the two field variables. Therefore solutions with two permutation branes can only be found in models where there exist two pairs of equal weights. These are: $\mathbb{P}_{(1,1,1,1,1)}[5]$, $\mathbb{P}_{(1,1,1,3,3)}[9]$, $\mathbb{P}_{(1,1,3,5,5)}[15]$ and $\mathbb{P}_{(1,3,3,7,7)}[21]$. Table 6 shows all simple tadpolefree configurations up to a permutation of variables. For the D-branes we use the notation $(d_1,\ldots,d_N)$ with $d_i=n_i$ for tensor product branes and $d_i=\{\eta_1,\eta_2,\ldots\}$, $\eta_j\in\mathcal{I}_i$ for permutation

Model	4 D-branes	Orientifold, $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$, $\epsilon = -1,\ M_\chi = 0,\ m_i = 0\ \forall i$
$\mathbb{P}_{(1,1,1,1,1)}[5]$	$M = 5,\ (3,3,3,3,3)$	$\sigma = \mathrm{id}$
	$M = 5,\ (3, \{\eta_2\}, \{\eta_2\})$	$\sigma = (2 \leftrightarrow 3, 4 \leftrightarrow 5)$
$\mathbb{P}_{(1,1,1,3,3)}[9]$	$M = 9,\ (5, \{\eta_4\}, \{\eta_1\})$	$\sigma = (2 \leftrightarrow 3, 4 \leftrightarrow 5)$
$\mathbb{P}_{(1,3,3,7,7)}[21]$	$M = 21,\ (11, \{\eta_3\}, \{\eta_1\})$	$\sigma = (2 \leftrightarrow 3, 4 \leftrightarrow 5)$

Table 7: Simple tadpolefree and spacetime-supersymmetric backgrounds

branes. In some of these configurations we have nontrivial residue factors in the charges: it is rewarding to see them to be matched exactly.

Using (54) we calculate the M-value for all configurations and check with (101) whether each configuration is parity-invariant. It turns out that this is only the case for four backgrounds, which are shown in table 7. Additionally we verify using (130) that these four configurations are spacetime-supersymmetric. Note that we always have four D-branes in each configuration due to the four external dimensions.

6.3 Constructing general backgrounds

Here we give detailed instructions how to construct general Fermat-type backgrounds which are tadpolefree and spacetime-supersymmetric. We suggest the following strategy:

1. Choose a model $\mathbb{P}_{(w_1,w_2,w_3,w_4,w_5)}[H]$
2. Choose a parity functor: $\mathcal{P}(\tau) = \mathscr{P}(\tau)$ or $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$. Because of the open issues in conjunction with $\mathcal{P}(\tau) = \mathscr{P}(\tau)$, which were discussed in subsection 6.1, it is advisable to work with $\mathcal{P}(\tau) = \mathscr{A}\mathscr{P}(\tau)$.
3. Choose values for ϵ, M_χ and m_i to fix the parity. Because of the problems discussed in the subsections 5.3.3 and 5.5 we suggest to set $M_\chi = 0,\ m_i = 0\ \forall i$. Furthermore choose a permutation σ for the parity.
4. By using (129) resp. (130) write down all combinations of M and N which lead to a supersymmetric configuration with the Orientifold.
5. Define a set of D-branes with the M-values determined above. Use (101) to mark the irreducible parity-invariant D-branes in the set. Make sure that the set includes the parity images of all non-invariant D-branes.

Define a new set with all irreducible parity-invariant D-branes and the direct sums of non-invariant D-branes and their parity images. This new set contains the building blocks of the string background to be constructed.

6. Use the rules (31) to write down all B-type Ramond-Ramond ground states $|\phi\rangle^l_{(R,R)} x_a^{t_a} x_b^{t_b} |0\rangle^l_{(R,R)}$.

7. Use (60) and (123) to calculate the charges of the Orientifold and the D-branes and write them as expansion in ω^{jl}, $j \in \{0, \dots, H-1\}$:

$$
\begin{aligned}
|I_{l,t} = 5| : \qquad & \mathrm{ch}(\mathcal{P}(\tau))(|\phi\rangle^l_{(R,R)}) = \sum_{j=0}^{H-1} p_j \omega^{jl} \\
& \mathrm{ch}(Q_i, \gamma_i)(|\phi\rangle^l_{(R,R)}) = \sum_{j=0}^{H-1} a_{ji} \omega^{jl} \\
|I_{l,t} = 3| : \qquad & \mathrm{ch}(\mathcal{P}(\tau))(|\phi\rangle^l_{(R,R)}) = \sum_{j=0}^{H-1} \widetilde{p}_j(t_a) \omega^{jl} \\
& \mathrm{ch}(Q_i, \gamma_i)(|\phi\rangle^l_{(R,R)}) = \sum_{j=0}^{H-1} \widetilde{a}_{ji}(t_a) \omega^{jl}
\end{aligned}
$$

8. Write down the following matrices and vectors:

$$
A = (a_{ji}) = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \end{pmatrix}
$$

$$
\widetilde{A}(t_a) = (\widetilde{a}_{ji}(t_a)) = \begin{pmatrix} \widetilde{a}_{11}(t_a) & \widetilde{a}_{12}(t_a) & \cdots \\ \widetilde{a}_{21}(t_a) & \widetilde{a}_{22}(t_a) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \widetilde{P} = \begin{pmatrix} \widetilde{p}_1(t_a) \\ \widetilde{p}_2(t_a) \\ \vdots \end{pmatrix}
$$

9. Solve the linear system of equations

$$
\frac{1}{D} AB = -P, \quad \frac{1}{D} \widetilde{A}(t_a) B = -\widetilde{P}(t_a), \quad B = (b_1, b_2, \dots)^T, \quad b_i \in \mathbb{N}_0 \ \ \forall i
$$

The numbers b_i to be calculated count the number of D-branes Q_i in the background. If there are no solutions, then enlarge the set of D-branes, if possible, and retry. Also have a look at [15] for some useful tools which help to solve these equations.

7 Massive Landau-Ginzburg Orbifolds

In section 4.9 we have made a comparison between the LG- and the CFT-description of D-branes. We have noted that D-branes described in the CFT-language sometimes have an additional ψ-label, which does not seem to be present in the LG formalism. In the following we shed some light on this issue.

Consider a Landau-Ginzburg orbifold with superpotential $W(x_i)$. This theory is equivalent to a *massive Landau-Ginzburg orbifold* with superpotential $\widetilde{W} = W(x_i) + \sum_i(-z_{i1}^2 - z_{i2}^2)$ [33]. In these models there exist some special types of reducible D-branes. To study these, let us first work with the superpotential $W = x^h - z^2$ and let us assume that h is even. Consider the following D-brane:

$$\begin{aligned} Q &= \begin{pmatrix} 0 & x^{h/2} \\ x^{h/2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} \\ \rho &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{132}$$

According to (82) and (84) this D-brane is reducible and equivalent to

$$\begin{aligned} \widehat{Q} = UQU^{-1} &= \begin{pmatrix} 0 & x^{h/2} + z \\ x^{h/2} - z & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x^{h/2} - z \\ x^{h/2} + z & 0 \end{pmatrix} \\ \widehat{\rho} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{133}$$

The transformation matrices U and U^{-1} are computed using (83):

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

We want to calculate the orbifold representation matrix associated to the matrix factorization (133). To do this we need to find out, how the orbifold group should act on the field z. In the models with H odd it is not possible to extend the $\mathbb{Z}_H$ orbifold group to the additional field, because its exponent is not a divisor of H. Therefore it seems to be natural to let the orbifold group act trivially on z. Then the orbifold representation matrix has the following form:

$$\gamma = \omega^{-M/2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \omega^{-M/2} \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

If we now transform this matrix using U and U^{-1} we get

$$\gamma = \omega^{-M/2} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

This matrix maps the irreducible components of $\widehat{Q}$ to itself, but with the opposite grading. These component D-branes are therefore not invariant under the orbifold group. The situation changes, if we consider the more general superpotential $\widetilde{W}$. Now we can have tensor products of N_S D-branes of type (132). If N_S is even, then the D-brane is invariant under the orbifold group. The irreducible component D-branes can be identified with the *short orbit branes* $\widehat{\mathscr{B}}_{k_i/2,M_i,S_i}$ described in [15]. The ψ-label is used in the CFT language whenever a D-brane splits up into two inequivalent D-branes containing short orbit branes. In principle we could use these D-branes to construct string backgrounds, but there is a problem which we already mentioned in section 4.6: the orbifold group treats the two fields associated to such a D-brane differently and it is not clear, whether the charge formula applies for this situation.

8 Summary

In this work we have shown that the Landau-Ginzburg formalism is well suitable to construct string backgrounds. D-branes and Orientifolds both fit nicely into mathematical structures, the former as objects in a triangulated category and the latter as functors in this category. The charges of both objects can be computed with little effort using the formulas presented in this work. The parity invariance of D-branes and the spacetime-supersymmetry of the whole configuration can be checked quickly with the relations developed in this thesis. We have presented a step-by-step guide explaining the construction of string backgrounds in the Landau-Ginzburg language. These procedures are suitable to be implemented in computer programs in order to find interesting configurations.

The comparison of the Landau-Ginzburg formalism with the conformal field theory description has revealed some incompatibilities, which currently imposes restrictions on the Orientifolds and D-branes to be used for constructing string backgrounds. The resolution of these problems will improve the applicability of the Landau-Ginzburg approach even more.

Acknowledgements

I want to thank Prof. Ilka Brunner for supervising this Master thesis. The fruitful discussions and the warm atmosphere were certainly the highlights during the half year. I am also grateful that I had the liberty of following my own paths while still being guided into the right direction.

References

[1] M. R. Douglas and S. Kachru, "Flux compactification," *Rev. Mod. Phys.* **79** (2007) 733–796, hep-th/0610102.

[2] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond,". Cambridge, UK: Univ. Pr. (1998) 531 p.

[3] J. Polchinski and Y. Cai, "Consistency of Open Superstring Theories," *Nucl. Phys.* **B296** (1988) 91.

[4] N. P. Warner, "N=2 supersymmetric integrable models and topological field theories," hep-th/9301088.

[5] D. Gepner, "Exactly Solvable String Compactifications on Manifolds of SU(N) Holonomy," *Phys. Lett.* **B199** (1987) 380–388.

[6] B. R. Greene, C. Vafa, and N. P. Warner, "Calabi-Yau Manifolds and Renormalization Group Flows," *Nucl. Phys.* **B324** (1989) 371.

[7] E. Witten, "Phases of N = 2 theories in two dimensions," *Nucl. Phys.* **B403** (1993) 159–222, hep-th/9301042.

[8] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," hep-th/0403166.

[9] C. Vafa, "Topological Landau-Ginzburg models," *Mod. Phys. Lett.* **A6** (1991) 337–346.

[10] A. Kapustin and Y. Li, "Topological correlators in Landau-Ginzburg models with boundaries," *Adv. Theor. Math. Phys.* **7** (2004) 727–749, hep-th/0305136.

[11] K. Hori and J. Walcher, "D-brane categories for orientifolds: The Landau-Ginzburg case," hep-th/0606179.

[12] M. R. Douglas, "D-branes, categories and N = 1 supersymmetry," *J. Math. Phys.* **42** (2001) 2818–2843, hep-th/0011017.

[13] A. Kapustin and Y. Li, "D-branes in Landau-Ginzburg models and algebraic geometry," *JHEP* **12** (2003) 005, hep-th/0210296.

[14] J. Walcher, "Stability of Landau-Ginzburg branes," *J. Math. Phys.* **46** (2005) 082305, hep-th/0412274.

[15] I. Brunner, K. Hori, K. Hosomichi, and J. Walcher, "Orientifolds of Gepner models," *JHEP* **02** (2007) 001, hep-th/0401137.

[16] W. Lerche, C. Vafa, and N. P. Warner, "Chiral Rings in N=2 Superconformal Theories," *Nucl. Phys.* **B324** (1989) 427.

[17] K. A. Intriligator and C. Vafa, "Landau-Ginzburg Orbifolds," *Nucl. Phys.* **B339** (1990) 95–120.

[18] B. R. Greene, "String theory on Calabi-Yau manifolds," hep-th/9702155.

[19] K. Hori *et al.*, "Mirror symmetry,". Providence, USA: AMS (2003) 929 p.

[20] E. Witten, "On the Landau-Ginzburg description of N=2 minimal models," *Int. J. Mod. Phys.* **A9** (1994) 4783–4800, hep-th/9304026.

[21] C. Vafa, "String Vacua and Orbifoldized L-G Models," *Mod. Phys. Lett.* **A4** (1989) 1169.

[22] M. Vonk, "A mini-course on topological strings," hep-th/0504147.

[23] A. Neitzke and C. Vafa, "Topological strings and their physical applications," hep-th/0410178.

[24] C. Caviezel, S. Fredenhagen, and M. R. Gaberdiel, "The RR charges of A-type Gepner models," *JHEP* **01** (2006) 111, hep-th/0511078.

[25] I. Brunner, M. Herbst, W. Lerche, and B. Scheuner, "Landau-Ginzburg realization of open string TFT," *JHEP* **11** (2006) 043, hep-th/0305133.

[26] J. Knapp, "D-Branes in Topological String Theory," 0709.2045.

[27] H. Jockers and W. Lerche, "Matrix Factorizations, D-Branes and their Deformations," *Nucl. Phys. Proc. Suppl.* **171** (2007) 196–214, 0708.0157.

[28] K. Hori and J. Walcher, "F-term equations near Gepner points," *JHEP* **01** (2005) 008, hep-th/0404196.

[29] I. Brunner and M. R. Gaberdiel, "Matrix factorisations and permutation branes," *JHEP* **07** (2005) 012, hep-th/0503207.

[30] S. K. Ashok, E. Dell'Aquila, and D.-E. Diaconescu, "Fractional branes in Landau-Ginzburg orbifolds," *Adv. Theor. Math. Phys.* **8** (2004) 461–513, hep-th/0401135.

[31] I. Brunner and K. Hori, "Orientifolds and mirror symmetry," *JHEP* **11** (2004) 005, hep-th/0303135.

[32] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string,". Cambridge, UK: Univ. Pr. (1998) 402 p.

[33] A. Kapustin and Y. Li, "D-branes in topological minimal models: The Landau- Ginzburg approach," *JHEP* **07** (2004) 045, hep-th/0306001.