COMPLEX ANALYSIS OF LINEAR ORDINARY DIFF. EQUATIONS

We consider the equation $\sum_{i=0}^{m} P_i(z)\, y^{(i)}(z) = 0$ and we assume $P_i(z)$ to be meromorphic functions with $P_m(z)=1$

The singularities of the solutions are related to the singularities of the coefficients $P_i(z)$.

More precisely, one can show that the only singularities in the solutions are those of the coefficients.

EXAMPLE

$y'' + \frac{1}{z} y' = 0$

$y = \text{const.}$ is a solution. The point $z=0$ is a singularity of the coefficient $\frac{1}{z}$ which is not a singularity of this solution

EXAMPLE

$y' + y^2 = 0$

$y = \frac{1}{z-a}$ is a solution and it is singular in $z=a$ while the coefficients are regular $\Rightarrow$ but the equation is not linear !

EXAMPLE

$y' - \frac{\alpha}{z} y = 0 \qquad \alpha \in \mathbb{R}$

the coefficient is singular at $z=0$. The equation is linear and $y = z^{\alpha}$ is a solution. Depending on the value of α it can have a pole, a branch point or it can be regular at $z=0$.

Regular singular point (fuchsian point)

Consider the equation $\sum_{i=0}^{m} P_i(z)\, y^{(i)}(z) = 0 \qquad P_m(z)=1$

A point $z=a$ is said to be a regular singular point if $P_{m-i}(z)$ has a pole at most of order i at $z=a$.

If we focus on the case of the second order equation we have

$$y'' + p(z)y' + q(z)y = 0 \quad \Rightarrow \quad \begin{cases} p(z) \text{ must have at most a } \underline{\text{single}} \text{ pole} \\ q(z) \quad " \quad " \quad \text{a } \underline{\text{double}} \text{ pole} \end{cases}$$

In a neighborhood of a regular singular point z_0 the functions $p(x)$ and $q(x)$ behave as

$$p(z) = \frac{a_{-1}}{z-z_0} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

$$q(z) = \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{z-z_0} + \sum_{k=0}^{\infty} b_k (z-z_0)^k$$

Introducing $P(z) \equiv (z-z_0)\, p(z)$ and $Q(z) = (z-z_0)^2 q(z)$ we have

$$P(z) = \sum_{k=0}^{\infty} a_{k-1} (z-z_0)^k \qquad Q(z) = \sum_{k=0}^{\infty} b_{k-2} (z-z_0)^k$$

If we multiply the differential equation by $(z-z_0)^2$ we obtain

$$(z-z_0)^2 y'' + (z-z_0) P(z)\, y' + Q(z)\, y = 0$$

We can now look for a solution by using the $\underline{\text{Frobenius method}}$

$$\boxed{y(z) = (z-z_0)^{\nu} \sum_{k=0}^{\infty} c_k (z-z_0)^k}$$

$c_0 = 1$ (irrelevant normalization)

ν to be determined

Plugging this ansatz in our equation we get

$$(z-z_0)^2 \underbrace{\left[\sum_{k=0}^{\infty} c_k (k+\nu)(k+\nu-1) (z-z_0)^{k+\nu-2} \right]}_{y''}$$

$$+ (z-z_0) \underbrace{\sum_{k=0}^{\infty} a_{k-1} (z-z_0)^k}_{P(z)} \underbrace{\sum_{\ell=0}^{\infty} c_\ell (\ell+\nu)(z-z_0)^{\ell+\nu-1}}_{y'} +$$

$$+ \underbrace{\sum_{k=0}^{\infty} b_{k-2} (z-z_0)^k}_{Q(z)} \underbrace{\sum_{\ell=0}^{\infty} c_\ell (z-z_0)^{\ell+\nu}}_{y} = 0$$

$$(z-z_0)^2 \left[\sum_{k=0}^{\infty} c_k (k+\nu)(k+\nu-1)(z-z_0)^{k+\nu-2} \right]$$

$$+ (z-z_0) \sum_{k=0}^{\infty} a_{k-1} (z-z_0)^k \sum_{\ell=0}^{\infty} c_\ell (\ell+\nu)(z-z_0)^{\ell+\nu-1} +$$

$$+ \sum_{k=0}^{\infty} b_{k-2} (z-z_0)^k \sum_{\ell=0}^{\infty} c_\ell (z-z_0)^{\ell+\nu} = 0$$

In the second and third term we send $k+\ell \to k'$: this way the sum over ℓ goes up to k'. We finally call $k' \to k$ and write

$$\sum_{k=0}^{\infty} (z-z_0)^{k+\nu} \left\{ c_k (k+\nu)(k+\nu-1) + \sum_{\ell=0}^{k} \left[(\ell+\nu) a_{k-1-\ell} + b_{k-2-\ell} \right] c_\ell \right\} = 0$$

from here we read the recursion relation

$$c_k (k+\nu)(k+\nu-1) = - \sum_{\ell=0}^{k} \left[(\ell+\nu) a_{k-1-\ell} + b_{k-2-\ell} \right] c_\ell$$

Setting $k=0$ we find ($c_0=1$)

$$\boxed{\nu(\nu-1) = -\nu a_{-1} - b_{-2}} \qquad \text{INDICIAL EQUATION}$$

which relates the possible values of ν to the coefficients a_{-1} and b_{-2}.

The roots of this equation are called characteristic exponents of z_0

Through this equation one finds the possible values of ν and then computes the coefficients $c_1, c_2, c_3 \dots$ by recursion.

If ν_1 and ν_2 are the characteristic exponents we can then state the following theorem.

THEOREM (without proof) (Fuchs)

If the differential equation $y'' + p(z)y' + q(z)y = 0$ has a regular singular point at $z = z_0$ then at least one solution can be written as

$$y_1(z) = (z-z_0)^{\nu_1} \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

where ν_1 is the characteristic exponent with the largest real part.

The second solution is either of this form, or if $\nu_1 - \nu_2 = n$ integer, it may take the form

$$y_2(z) = (z-z_0)^{\nu_2} \sum_{k=0}^{\infty} d_k (z-z_0)^k + d\, y_1(z) \ln(z-z_0)$$

additional logarithmic term

EXAMPLE : BESSEL EQUATION

We have already encountered the Bessel equation

$$y'' + \frac{1}{z} y' + \left(1 - \frac{\alpha^2}{z^2}\right) y = 0$$

in this case $p(z) \sim \frac{1}{z}$ $\quad q(z) \sim \frac{1}{z^2}$

$\Rightarrow$ $z = 0$ is a regular singular point

$a_{-1} = 1 \qquad b_{-2} = -\alpha^2$

$\Rightarrow$ the indicial equation is

$$\nu(\nu-1) = -\nu + \alpha^2$$
$$\nu^2 = \alpha^2$$

and the solutions are

$$\nu_1 = \alpha \qquad \nu_2 = -\alpha$$

$\Rightarrow$ we conclude that there are two linearly independent solutions provided that 2α is not an integer

EXAMPLE : RADIAL EQUATION IN COULOMB POTENTIAL

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{\psi}{r^2}\,\ell(\ell+1) + \frac{2m}{\hbar^2}\Big(V(r)-E\Big)\psi = 0$$

$$\psi'' + \frac{2}{r}\psi' + \Big[\underbrace{\frac{2m}{\hbar^2}\big(V(r)-E\big)}_{\hookrightarrow\, O\left(\frac{1}{r}\right)} - \frac{\ell(\ell+1)}{r^2}\Big]\psi = 0$$

$\Rightarrow$ $r=0$ is a regular singular point for which $a_{-1}=2$ and $b_{-2}=-(\ell+1)\ell$

The indicial equation is

$$\nu(\nu-1) = -\nu a_{-1} - b_{-2} = -2\nu + \ell(\ell+1)$$

with solutions $\qquad \nu=\ell \qquad \nu=-1-\ell$

They differ by an integer! $\Rightarrow$ one solution is of the form $\displaystyle r^\ell\sum_{k=0}^{\infty} c_k r^k$

EXAMPLE : HYPERGEOMETRIC EQUATION (more later)

$$y'' + \frac{\gamma-(\alpha+\beta+1)z}{z(1-z)}\,y' - \frac{\alpha\beta}{z(1-z)}\,y = 0$$

A large number of functions in mathematical physics are solutions of this equation for appropriate choices of α, β, γ. The regular singular points are $z=0$ and $z=1$.

At $z=0$ we have $a_{-1}=\gamma \quad b_{-2}=0 \quad\Rightarrow$ the indicial equation is

$$\nu(\nu-1) = -\gamma\nu$$

with roots $\nu_1=0 \quad \nu_2=1-\gamma \quad\Rightarrow$ we have two solutions provided that γ is NOT an integer

FUCHSIAN DIFFERENTIAL EQUATIONS

In many physically interesting cases the behavior of the solution of a differential equation at infinity matters. For example, to describe a bound state as solution of the Schroedinger equation, it is important that the wave function vanishes at large distances. Let us consider the equation

$$y'' + p(z)\,y' + q(z)\,y = 0$$

To study the behavior at infinity, we can set $z = \frac{1}{t}$ and investigate the behavior at $t=0$. We define $v(t) \equiv y\left(\frac{1}{t}\right)$ and we find

$$v'(t) = y'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)$$

$$v''(t) = y''\left(\frac{1}{t}\right)\frac{1}{t^4} + y'\left(\frac{1}{t}\right)\frac{2}{t^3}$$

$$\Rightarrow \quad y'' = t^4 v'' - 2t y' = t^4 v'' + 2t^3 v'$$

$$y' = -t^2 v'$$

The differential equation becomes

$$v'' + \frac{2}{t}v' - \frac{p\left(\frac{1}{t}\right)}{t^2}v' + \frac{q\left(\frac{1}{t}\right)}{t^4}v = 0$$

We define $\tilde{p}(t) \equiv p\left(\frac{1}{t}\right)$ and $\tilde{q}(t) \equiv q\left(\frac{1}{t}\right)$

If ∞ must be a regular singular point for the original equation $t=0$ must be a regular singular point for the new equation $\Rightarrow$ we must have

$$\tilde{p}(t) = a_1 t + a_2 t^2 + \dots \qquad \tilde{q}(t) = b_2 t^2 + b_3 t^3 + \dots$$

$$\Rightarrow \quad p(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \qquad q(z) = \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots$$

In this case we can invoke the Fuchs theorem and say that a solution exists as

$$y_1(z) = z^{-d}\sum_{k=0}^{\infty} c_k z^{-k}$$

DEF: A linear homogeneous differential equation in the complex domain of the form

$$\sum_{i=0}^{m} P_i(z)\, y^{(i)}(z) = 0 \qquad P_m(z) = 1$$

with meromorphic coefficients is called FUCHSIAN if it has only regular singular points in the extended complex plane (that is, on $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$)

We will see that a particular kind of Fuchsian equation describes a large class of non elementary functions we encounter in mathematical physics.

Consider the equation $\quad y'' + p(z)\, y' + q(z)\, y = 0$

in the case in which there are at most two regular singular points.

We call these points z_1 and z_2 and we introduce a new variable $\xi(z) = \dfrac{z - z_1}{z - z_2}$

This (Mobius) transformation maps the two regular singular points

in $\xi_1 = \xi(z_1) = 0$ and $\xi_2 = \xi(z_2) = \infty$ in $\bar{\mathbb{C}}$.

Since the transformation is an invertible mapping we can define $u(\xi) \equiv y(z(\xi))$

and we have $\quad u'(\xi) = y'(z)\, z' \qquad u''(\xi) = y''(z)\, z'^2 + y'(z)\, z''$

$$\Rightarrow \quad y'' = \frac{u'' - y' z''}{z'^2} \qquad y' = \frac{u'}{z'}$$

NOTE THAT

$$z' = \frac{z_1 - z_2}{(1-\xi)^2}$$

$$\frac{z''}{z'} = \frac{2}{1-\xi}$$

and the original equation becomes

$$\frac{u'' - u' \dfrac{z''}{z'}}{z'^2} + p(z)\, \frac{u'}{z'} + q(z)\, u = 0$$

or

$$u'' + \overbrace{\left(p(z)\, z' - \frac{z''}{z'} \right)}^{\varphi(\xi)} u' + \overbrace{q(z)\, z'^2}^{\vartheta(\xi)}\, u = 0$$

So the equation takes the form

$$u'' + \varphi(\xi)\, u' + \theta(\xi) u = 0$$

where now $\xi = 0$ and $\xi = \infty$ are the regular singular points.

But the fact that $\underline{\xi = 0}$ is a regular singular point implies that

$$\varphi(\xi) = \frac{a_{-1}}{\xi} + a_0 + a_1 \xi + \cdots \qquad \theta(\xi) = \frac{b_{-2}}{\xi^2} + \frac{b_{-1}}{\xi} + b_0 + \cdots$$

On the other hand, the fact that $\underline{\xi = \infty}$ is a regular singular point implies that

$$\varphi(\xi) = \frac{a_{-1}}{\xi} + \frac{a_{-2}}{\xi^2} + \cdots \qquad \theta(\xi) = \frac{b_{-2}}{\xi^2} + \frac{b_{-3}}{\xi^3} + \cdots$$

By combining these two requirements we conclude that

$$\boxed{\varphi(\xi) = \frac{a_{-1}}{\xi}} \qquad \boxed{\theta(\xi) = \frac{b_{-2}}{\xi^2}}$$

$\Rightarrow$ When there are only two regular singular points the equation is equivalent to an Euler equation, which can be transformed to an equation with constant coefficients, and produces nothing new (no special functions!)

(indeed we get $\xi^2 u'' + a_{-1} \xi u' + b_{-2} u = 0 \Rightarrow$ usual ansatz $u = \xi^\alpha$)

$\Rightarrow$ The simplest non-trivial case is the one in which there are $\underline{\text{3 regular singular points}}$

A general second-order Fuchsian differential equation with three regular-singular points has the form

$$y'' + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right] y'$$

$$+ \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right] \frac{y}{(z-a)(z-b)(z-c)} = 0$$

where a, b, c are the three regular singular points and α, α', β, β' and γ, γ' are their characteristic exponents (e.g.: $z = a \Rightarrow \nu(\nu-1) = -\nu a_{-1} - b_{-2}$ with $a_1 = 1-\alpha-\alpha'$ and $b_{-2} = \alpha\alpha' \Rightarrow \nu_1 = \alpha$ and $\nu_2 = \alpha'$)

This equation is called <u>RIEMANN DIFFERENTIAL EQUATION</u>

The Mobius transformation $\xi(z) = \dfrac{(z-a)(c-b)}{(z-b)(c-a)}$ maps a, b and c into $0, \infty, 1$

The solutions of the Riemann equation are usually denoted by the Pepperitz symbol (P-symbol)

$$y(z) = P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

The HYPERGEOMETRIC FUNCTION can be expressed as

$${}_2F_1(a,b,c;z) = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$$

A general solution of the Riemann equation can be expressed in terms of the Hypergeometric function as

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = \left(\frac{z-a}{z-b}\right)^{\alpha} \left(\frac{z-c}{z-b}\right)^{\gamma} P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \alpha+\beta+\gamma & 0 & \xi \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{matrix}\right\} \qquad \xi = \frac{(z-a)(c-b)}{(z-b)(c-a)}$$

Equivalently we can write

$$y(z) = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} {}_2F_1\left(\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma,\ 1+\alpha-\alpha',\ \frac{(z-a)(c-b)}{(z-b)(c-a)}\right)$$

$\Rightarrow$ we can conclude that the Riemann equation is equivalent to the Hypergeometric equation

HYPERGEOMETRIC FUNCTION

If we take the Riemann equation

$$y'' + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right] y'$$

$$+ \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right] \frac{y}{(z-a)(z-b)(z-c)} = 0$$

Corresponding to a solution $P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$

and we specialize to the case $a \to 0$, $b \to \infty$, $c \to 1$ by then doing the remapping $\alpha \to 0$, $\beta \to a$, $\gamma \to 0$, $\alpha' \to 1-c$, $\beta' \to b$, $\gamma' \to c-a-b$

The second term in the coefficient of y' vanishes, and the same happens for the first and third terms in the coefficient of y.

$\Rightarrow$ the equation takes the form

$$y'' + \left[\frac{c}{z} + \frac{1+a+b-c}{z-1}\right] y' + \frac{ab}{z(z-1)}\, y = 0$$

with solution $P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$

which can also be written as

$$\boxed{z(1-z)\, y'' + \left[c - (1+a+b)z\right] y' - ab\, y = 0}$$

HYPERGEOMETRIC EQUATION

This equation describes a large class of non-elementary functions in mathematical physics

We now look for a series solution of the equation by using the Frobenius method.

The characteristic exponents at $z=0$ are 0 and $1-c$

$\Rightarrow$ we look for an analytic solution of the form

$${}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty} c_k z^k \qquad c_0 = 1$$

we have

$$y' = \sum_{k=1}^{\infty} k c_k z^{k-1} \qquad y'' = \sum_{k=2}^{\infty} k(k-1) c_k z^{k-2}$$

Plugging this ansatz in the equation we obtain

$$\sum_{k=2}^{\infty} k(k-1) c_k \left(z^{k-1} - z^k\right) + c\sum_{k=1}^{\infty} k c_k z^{k-1} - (1+a+b)\sum_{k=1}^{\infty} k c_k z^k - ab\sum_{k=0}^{\infty} c_k z^k = 0$$

By relabelling the index $k \to k+1$ in the first and third term so as to factor out a common z^k term we find

$$k(k+1) c_{k+1} - k(k-1) c_k + c(k+1) c_{k+1} - (1+a+b) k c_k - ab\, c_k = 0$$

which implies

$$c_{k+1} = \frac{(a+k)(b+k)}{(k+1)(k+c)} c_k$$

If c is non zero and not a negative integer number the coefficients can be computed recursively

$$c_0 = 1 \qquad c_1 = \frac{ab}{c} \qquad c_2 = \frac{ab}{c}\frac{(a+1)(b+1)}{2(c+1)} = \frac{1}{2}\frac{a(a+1)\, b(b+1)}{c(c+1)}$$

$$c_3 = \frac{1}{3\cdot 2}\frac{(a+2)(a+1)a\ (b+2)(b+1)b}{(c+2)(c+1)c} \qquad \Rightarrow \quad c_k = \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\,\Gamma(k+1)}\ \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}$$

$\Rightarrow$ we can write

$$\boxed{{}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b+k)}{\Gamma(c+k)\,\Gamma(k+1)}\, z^k}$$

HYPERGEOMETRIC SERIES

$c \neq 0, -1, -2 \ldots$

The series is called hypergeometric because it generalises the case of ${}_2F_1(1,b,b,z) = (1-z)^{-1}$ which is the ordinary geometric series.

Second solution

We have seen that the characteristic indices are 0 and $1-c$. Let us assume $c \neq 1$ and replace $y = z^{1-c}u$ in the differential equation. We find

$$z(1-z)\,u'' + \big(2-c + z(-3-a-b+2c)\big)\,u' + (1+b-c)(c-a-1)\,u = 0$$

which corresponds to a new hypergeometric equation with

$c' = 2-c$

$a' = a+1-c$ $\Rightarrow$ a second solution of the hypergeometric equation

$b' = 1+b-c$ is $z^{1-c}\,{}_2F_1(a+1-c,\, 1+b-c,\, 2-c;\, z)$

The two solutions form a canonical basis of solutions of the differential equation at $z=0$.

It is possible to obtain many relations among the hypergeometric functions. There are 24 different (but linearly dependent) solutions (KUMMER'S SOLUTIONS)

Let us go back to the hypergeometric series

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(k+1)} z^k $$

A symmetry of the hypergeometric function that originates from the differential equation $(a \leftrightarrow b)$ is

$$ {}_2F_1(a,b,c;z) = {}_2F_1(b,a,c;z) $$

We also note that the series becomes a polynomial if either a or b are negative integers. Suppose a is a negative integer $\Rightarrow$ for $k < |a|$ both $\Gamma(a+k)$ and $\Gamma(a)$ have poles that cancel each other $\Rightarrow$ since $\Gamma(a+k)$ becomes finite for $k > |a|$, $\Gamma(a)$ remains divergent and the coefficient vanishes. The same reasoning works for b.

Many functions of mathematical physics are related to the hypergeometric. Even some elementary functions can be expressed in terms of the hypergeometric.

- For example

$$ {}_2F_1(a,b,b;z) = \sum_{k=0}^{\infty} \frac{\cancel{\Gamma(b)}}{\Gamma(a)\cancel{\Gamma(b)}} \frac{\Gamma(a+k)\cancel{\Gamma(b+k)}}{\cancel{\Gamma(b+k)}\Gamma(k+1)} z^k = (1-z)^{-a} $$

$$ = 1 + za + \frac{1}{2}(a+1)a z^2 + \dots $$

$$ {}_2F_1(1,1,2;-z) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \sum_{k=0}^{\infty} \frac{\Gamma(1+k)\cancel{\Gamma(1+k)}}{\Gamma(2+k)\cancel{\Gamma(1+k)}} (-1)^k z^k = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k+1} = \frac{\ln(1+z)}{z} $$

- Legendre Polynomials

$$ P_n(z) = {}_2F_1\left(-n, n+1, 1, \frac{1-z}{2}\right) $$

$$ P_0(z) = 1 $$
$$ P_1(z) = z $$
$$ P_2(z) = \frac{1}{2}(3z^2 - 1) $$
$$ \dots $$

We have defined the hypergeometric function through the series

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b+k)}{\Gamma(c+k)\,\Gamma(k+1)} z^k $$

but we have not discussed the convergence radius of the series.

By computing the ratio of the k+1 over the k term in the series we get

$$ \left| \frac{\dfrac{\Gamma(a+k+1)\Gamma(b+k+1)}{\Gamma(c+k+1)\Gamma(k+2)}}{\dfrac{\Gamma(a+k)\ \Gamma(b+k)}{\Gamma(c+k)\ \Gamma(k+1)}} z \right| = \left| \frac{(a+k)(b+k)}{(c+k)\ (k+1)} \right| |z| $$

$\Rightarrow$ through the RATIO TEST we can conclude that $R = 1$

It is possible to analytically continue the function outside the disk $|z|<1$

Indeed the Hypergeometric function admits the following <u>FUNDAMENTAL INTEGRAL REPRESENTATION</u>

$$ \boxed{ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt } $$

for $|z|<1$ we can prove this equation by expanding $(1-tz)^{-a}$ as $z \to 0$

$$ (1-zt)^{-a} = 1 + atz + \tfrac{1}{2}a(a+1)t^2z^2 + \cdots\cdots $$

By defining the Euler beta function $\quad B(x,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$

we have

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \left[B(b,c-b) + az\,B(b+1,c-b) + \tfrac{1}{2}a(a+1)z^2 B(b+2,c-b) + \cdots \right] $$

but $\quad B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$\Rightarrow \quad {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)} \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} + z\,\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(c+1)} + \right.$$

$$\left. + \frac{\Gamma(a+2)\Gamma(b+2)}{\Gamma(c+2)}\,\frac{z^2}{2} + \cdots \right]$$

and we recover the hypergeometric series.