DIRECT PRODUCT REPRESENTATIONS AND CLEBSCH - GORDAN COEFFICIENTS

Direct product space

Let U, V be vector spaces with inner product, and $\{u_i\}$, $\{v_j\}$ orthonormal bases for U and V, respectively. The direct product space is the space $W = U \times V$ consisting of all the linear combinations of vectors w_k, $k = (i,j)$ where w_k is the formal product of u_i and v_j.

EXAMPLE

Let $\underline{x}_1$ be the coordinate vector of particle 1 and $\underline{x}_2$ the coordinate vector of particle 2. The vector $|\psi_1(\underline{x}_1)\rangle$ is a vector in the Hilbert space H_1 describing the state of particle 1 while $|\psi_2(\underline{x}_2)\rangle$ is the corresponding vector in the Hilbert space H_2 for particle 2.
The state of the two-particle system is a vector in the direct product space $H_1 \times H_2$

Direct product representation

Let G be a symmetry group. Suppose $D^{\mu}(G)$ and $D^{\nu}(G)$ are two irreducible representations of G on the vector spaces U and V, respectively.

$\Rightarrow$ $D^{\mu\times\nu}(g) = D^{\mu}(g) \times D^{\nu}(g)$ on $W = U \times V$ and $g \in G$ also forms a representation, which is called <u>direct product representation</u> of D^{μ} and D^{ν}.

The character of $D^{\mu\times\nu}$ is simply the product of the two characters

$$\chi^{\mu\times\nu} = \chi^{\mu}\times\chi^{\nu}$$

Indeed $\chi^{\mu\times\nu} = \mathrm{Tr}\, D^{\mu\times\nu}(g) = D^{\mu\times\nu}(g)^{k}{}_{k} = D^{\mu}(g)^{i}{}_{i}\, D^{\nu}(g)^{j}{}_{j} = \chi^{\mu}\cdot\chi^{\nu}$

The dimension of the direct product representation is given by $n_\mu \times n_\nu$, i.e. the product of the dimensions of the two representations.

The direct product representation is usually reducible. The number of times the irreducible representations occur in the reduction is related to the Clebsch-Gordan coefficients and it can be computed by using the characters, with the theorem we have studied before.

EXAMPLE

Let us consider the product representations of the symmetric group S_3.

We know that there are 3 irreducible representations, with character table:

$\mu \backslash i$	1	2	3
1	1	1	1
2	1	-1	1
3	2	0	-1

Let us start with $D^1\times D^1$. We have $\chi^{1\times 1} = (1,1,1)$ $\Rightarrow D^{1\times 1}\sim D^1$. Analogously $D^{1\times 2}$ is such that $\chi^{1\times 2} = (1,-1,1)$ and thus $D^{1\times 2}\sim D^2$. Similarly we have $D^{2\times 2}\sim D^1$.

What about $D^{3\times 3}$? The dimension of D_3 is 2, so $D^{3\times 3}$ is 4-dimensional and must be reducible. We have $\chi^{3\times 3} = (4,0,1)$ and $D^{3\times 3} = e_1 D_1 \oplus e_2 D_2 \oplus e_3 D_3$

$$e_1 = \sum_i \chi_i^{1}\, \chi_i^{3\times 3}\, \frac{m_i}{m_G} = \frac{4}{6}+\frac{2}{6} = 1$$

$$e_2 = \sum_i \chi_i^{2}\, \chi_i^{3\times 3}\, \frac{m_i}{m_G} = 1$$

$$e_3 = \sum_i \chi_i^{3}\, \chi_i^{3\times 3}\, \frac{m_i}{m_G} = \frac{8}{6}-\frac{2}{6} = 1$$

dimensions

$$\Rightarrow D^{3\times 3} = \underset{1}{D_1}\oplus \underset{1}{D_2}\oplus \underset{2}{D_3}$$

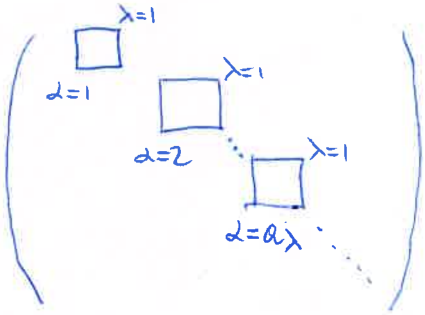
CLEBSCH-GORDAN COEFFICIENTS

The direct product representation is generally reducible and $D^{\mu\times\nu} = \bigoplus a_\lambda D^\lambda$

This means that the vector space $W = U \times V$ can be decomposed into the direct sum of invariant subspaces, corresponding to each of the irreducible representations. We can choose a basis of the form $w^\lambda_{\alpha e}$ where λ runs over the representations, $\alpha = 1 \dots a_\lambda$ and $e = 1 \dots m_\lambda$

↳ λ invariant subspace

spaces corresponding to the same λ



The orthonormal vectors $w^\lambda_{\alpha e}$ are related to the original vectors e_i, e_j by a unitary transformation

$$|w^\lambda_{\alpha e}\rangle = \sum_{ij} |w_{ij}\rangle \langle ij \mid \alpha, \lambda, e\rangle$$

↑ Clebsch-Gordan coefficients

EXAMPLE

System of 2 spin $\frac{1}{2}$ particles: total spin can be 1 or 0 $\Rightarrow$ $D^{\frac{1}{2}\times\frac{1}{2}} = D^1 \oplus D^0$

$$|\psi_1\rangle = \alpha\,|+\rangle_1 + \beta\,|-\rangle_1$$

$$|\psi_2\rangle = \alpha'\,|+\rangle_2 + \beta'\,|-\rangle_2$$

$$\Rightarrow \quad D^1 \begin{cases} |11\rangle = |+\rangle_1 |+\rangle_2 \qquad |1{-1}\rangle = |-\rangle_1 |-\rangle_2 \\ |10\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2\right) \end{cases}$$

$$D^0 \left\{ |00\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2\right) \right.$$

$D^{\frac{1}{2}}$ 2 dimensional

D^0 1 dimensional

D^1 3 dimensional

$2 \times 2 = 3 + 1$ ok!

Irreducible basis vectors

Let $U(G)$ a representation of a group G on an inner-product space V (we assume it to be unitary) and V_μ an invariant subspace with respect to $U(G)$.
The invariant subspace corresponds to an irreducible representation μ of G.
We call the set of basis vectors $\{ e_i^\mu, i = 1 \dots m_\mu \}$ of V_μ, which transforms as

$$U(g) | e_i^\mu \rangle = e_j^\mu D^\mu(g)^j{}_i \qquad g \in G$$

an IRREDUCIBLE set transforming according to the μ representation.

THEOREM

Let $\{ u_i^\mu, i = 1 \dots m_\mu \}$ and $\{ v_j^\nu, j = 1 \dots m_\nu \}$ be two sets of irreducible basis vectors with respect to the group G in a vector space V.

$\Rightarrow$ if μ and ν are not equivalent then the two invariant subspaces are orthogonal

Indeed

$$\langle v_\nu^j | u_i^\mu \rangle = \langle v^j_\nu | U^\dagger(g) U(g) | u_i^\mu \rangle$$

$$= D_\nu^\dagger(g)^j{}_k D^\mu(g)^\ell{}_i \langle v_\nu^k | u_\ell^\mu \rangle$$

$$= \sum_g D_\nu^\dagger(g)^j{}_k D^{\mu\,\ell}{}_i(g) \cdot \frac{1}{m_G} \langle v_\nu{}^k | u_\ell^\mu \rangle$$

since it is independent on g we can sum over g and divide by m_G

$$= \frac{1}{m_\mu} \delta^\mu{}_\nu \, \delta^j{}_i \, \delta^k{}_\ell \langle v_\nu^k | u_\ell^\mu \rangle$$

$$= \frac{1}{m_\mu} \delta^\mu{}_\nu \, \delta^j{}_i \langle v_\nu^k | u_k^\mu \rangle$$

$\Rightarrow$ all vectors in one invariant subspace are orthogonal to all vectors in the other

EXAMPLE: Hydrogen atom: states corresponding to different angular momenta (corresponding to different representations of the rotation group) are always orthogonal!

Irreducible Operators and the Wigner-Eckart Theorem

Under symmetry group transformations, the operators acting on the vector space of the physical solutions also transform in a definite way. They are naturally classified according to the irreducible representations of the symmetry group.

Irreducible operators

A set of operators $\{O_i^\mu, i = 1 \cdots n_\mu\}$ on a vector space V transforming as

$$U(g)\, O^\mu_i\, U(g)^{-1} = O^\mu_j\, D^\mu(g)^j{}_i$$

is said to be an irreducible set of operators corresponding to the μ representation.

These operators are also sometimes called IRREDUCIBLE TENSORS

Given $\{O^\mu_i\}$ and $\{e^\nu_j\}$ irreducible operators and vectors we can ask ourselves how $O^\mu_i |e^\nu_j\rangle$ transforms. We have

$$U(g)\, O^\mu_i\, |e^\nu_j\rangle = U(g)\, O^\mu_i\, U(g)^{-1} U(g)\, |e^\nu_j\rangle$$

$$= O^\mu_k |e^\nu_l\rangle\, D^\mu(g)^k{}_i\; D^\nu(g)^l{}_j$$

$\Rightarrow$ these states transform according to the DIRECT PRODUCT representation $\mu \times \nu$

We can then expand the vectors $O^\mu_i |e^\nu_j\rangle$ in terms of the irreducible vectors $|w^\lambda_{\alpha l}\rangle$ using the Clebsch-Gordan coefficients

$$O^\mu_i |e^\nu_j\rangle = \sum_{\alpha,\lambda,l} |w^\lambda_{\alpha l}\rangle \langle \alpha, \lambda, l\, (\mu,\nu)\, i j\rangle$$

$\Rightarrow$ by multiplying by $\langle e^l_\lambda|$ on the left we obtain

$$\boxed{\langle e^{\ell}_{\lambda} | O^{\mu}_{i} | e^{\nu}_{j} \rangle = \sum_{\alpha} \langle \alpha, \lambda, \ell (\mu, \nu)\, i, j \rangle \langle \lambda | O^{\mu} | \nu \rangle_{\alpha}}$$

where $\langle \lambda | O^{\mu} | \nu \rangle_{\alpha} = \frac{1}{m_{\lambda}} \sum_{k} \langle e^{k}_{\lambda} | w^{\lambda}_{\alpha k} \rangle$ is called reduced matrix element

This result goes under the name of Wigner-Eckart theorem.

It basically tells us that the multitude of matrix elements on the left-hand side is determined by a few, reduced matrix elements that do not depend on the indices ℓ, i, j and on Clebsch-Gordan coefficients, specified by group representation theory.

There are several applications of this important theorem.

<u>EXAMPLE</u> (more in QM1)

Electromagnetic transitions in atoms. Since electromagnetism is invariant under rotations, the symmetry group is $SO(3)$. Electromagnetic transitions involve the emission of a photon while the atom jumps from an initial state to a final state. The probability for such (multipole) transition is controlled by a matrix element $\langle j', m' | O^{S}_{x} | j m \rangle$ that can be evaluated using the above theorem

j': final state

O^{S}_{x}: multipole operator

$j m$: initial state

<u>EXAMPLE</u> (more later)

Study of pion-nucleon cross sections. Here the symmetry group is NUCLEAR ISOSPIN

CONTINUOUS GROUPS

Continuous groups consist of elements which are labelled by one or more continuous parameters (say $\theta_1, \theta_2 .. \theta_m$) where each variable has a well defined range. Clearly a continuous group must have an infinite number of elements. Note, however, that the reverse is not true : a counter example is given by translations of an infinite lattice (so, not all the infinite groups are continuous). The general mathematical theory of continuous groups is usually called theory of LIE GROUPS.

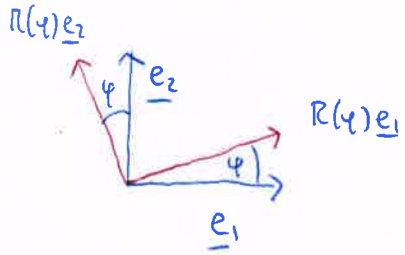
LIE GROUP ~ infinite group whose elements can be parametrized analytically in a smooth way

A precise definition of Lie group requires concepts of topology and differential geometry beyond our course. If a group has only one parameter, we can assume if it is a Lie group $g = g(\theta)$ with $g(0) = e$. Elements "close" to the identity can be obtained with "small" values of θ.

THE ROTATION GROUP $SO(2)$ (1-parameter Lie group)

We consider a system symmetric under rotations around a point O in the plane and we adopt a Cartesian system with $\underline{e_1}$ and $\underline{e_2}$ as basis vectors. Denoting the rotation with an angle φ by $R(\varphi)$ we have

$$\begin{cases} R(\varphi)\underline{e_1} = \underline{e_1}\cos\varphi + \underline{e_2}\sin\varphi \\ R(\varphi)\underline{e_2} = -\underline{e_1}\sin\varphi + \underline{e_2}\cos\varphi \end{cases}$$



or equivalently $R(\varphi)\underline{e_i} = R(\varphi)^j{}_i\,\underline{e_j}$ (j: row, i: column)

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

If $\underline{x}$ is a vector in the plane $\underline{x} = (x_1, x_2)$ in the basis of $\underline{e_1}$ and $\underline{e_2}$ we have

$$\underline{x} \to R\underline{x} = R(\varphi)\underline{e_i}\,x^i = R(\varphi)^j{}_i\,\underline{e_j}\,x^i$$

sum over repeated indices is understood

$\Rightarrow$ this implies that $(x')^j = R(\varphi)^j{}_i\,x^i$

The transformation structure is clearly the one of a group:

$$R(\varphi_1+\varphi_2) = R(\varphi_1)R(\varphi_2) = R(\varphi_2)R(\varphi_1)$$

abelian group

$$R(\varphi) = R(\varphi \pm 2\pi) \qquad R(0) = E \qquad R(-\varphi) = R^{-1}(\varphi)$$

global property

Geometrically it is obvious that a rotation leaves the length of a vector invariant $\Rightarrow$ from $x^j \to R(\varphi)^j{}_i\,x^i$ we obtain

$$|x|^2 = x^i x^j \delta_{ij} \to R(\varphi)^i{}_k\,x^k\,R(\varphi)^j{}_l\,x^l\,\delta_{ij} = \underbrace{R(\varphi)^i{}_k\,R(\varphi)^i{}_l}\,x^k x^l$$

must be equal to δ_{kl} to leave $|x|^2$ invariant

$\Rightarrow$ $R^T(\varphi)R(\varphi) = E$ is the necessary condition

Matrices with this property are called <u>orthogonal</u>.

Such matrices are required to have $\det^2(R) = 1$ which means $\det R = \pm 1$. The condition that the relation can be obtained smoothly from the identity requires (since $\det E = 1$) the more restrictive condition $\det R = 1$.

The 2-dimensional rotations form a group called $SO(2)$ ← orthogonal

← "special" orthogonal transformations

The requirement that $\det R = +1$ excludes reflections

(NOTE THAT $R = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ is a rotation! $(\varphi = \pi)$ in 2 dimensions rotations are represented by $R = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ or $R = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ and have $\det R = -1$)

The generator of SO(2)

Although $SO(2)$ has an infinite number of elements, the structure of the group is almost completely determined by the multiplication rule and continuity. Consider an infinitesimal rotation by an angle $d\varphi$. Differentiability of $R(\varphi)$ requires that $R(d\varphi)$ must differ from E by a quantity of order $d\varphi$.

$$\Rightarrow \quad R(d\varphi) = E - \underbrace{i}_{\text{added by convention for later convenience}} d\varphi \, J$$

We now consider $R(\varphi + d\varphi)$. We can evaluate it in two ways

$$R(\varphi + d\varphi) = \underbrace{R(\varphi) R(d\varphi)}_{\text{abelian group}} = R(\varphi) - i\, d\varphi \, R(\varphi) J$$

$$R(\varphi + d\varphi) = R(\varphi) + d\varphi \frac{dR}{d\varphi}$$

$\Rightarrow$ $$\frac{dR}{d\varphi} = -i\, R(\varphi)\, J$$ differential equation for $R(\varphi)$ with the boundary condition $R(0) = E$

$\Rightarrow$ the solution is $R(\varphi) = e^{-i\varphi J}$

All rotations in the plane can be expressed as $e^{-i\varphi J}$ where J is called GENERATOR of $SO(2)$.

We can thus say that the group structure and its representations are determined by a simple generator J which is specified by the behavior of the rotation around the identity.

This is typical of the power (and beauty) of Lie groups:
$\Rightarrow$ the local behavior of the group near the identity determines the most important properties of these groups. Once J is known, the group elements can be obtained.

But it is important to note that GLOBAL properties cannot be determined from the generator. For example $R(\varphi + 2\pi) = R(\varphi)$, which is of TOPOLOGICAL origin, plays a role in determining the irreducible representations of $SO(2)$.

Let us focus on the explicit representation provided by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

we have
$$R(d\varphi) \approx \begin{pmatrix} 1 & -d\varphi \\ d\varphi & 1 \end{pmatrix} = E - i\, d\varphi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\Rightarrow$ $J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ a traceless hermitian matrix with $J^2 = E \quad J^3 = J \ldots$

$$e^{-iJ\varphi} = E - iJ\varphi - \frac{1}{2}E\varphi^2 + \cdots = E\cos\varphi - iJ\sin\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

IRREDUCIBLE REPRESENTATIONS OF SO(2)

Consider a representation of $SO(2)$ on a finite dimensional vector space and call $U(\varphi)$ the operator corresponding to $R(\varphi)$. We must have

$$U(d\varphi) = E - i\, d\varphi\, J_U$$

where J_U is the generator of the group in the representation U. Repeating the same arguments as before we obtain

$$U(\varphi) = e^{-i\varphi J_U} \qquad U \text{ unitary} \Rightarrow J_U \text{ hermitian}$$

Since $SO(2)$ is abelian $\Rightarrow$ all irreducible representations are one-dimensional.

If $|\alpha\rangle$ is a vector in one of the irreducible representations, we must have

$$J|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha \text{ real number}$$

eigenvalue of J

and thus

$$U(\varphi)|\alpha\rangle = |\alpha\rangle e^{-i\varphi\alpha}$$

This representation automatically fulfills the group multiplication rule.

However, in order to satisfy the global constraint $R(\varphi \pm 2\pi) = R(\varphi)$ α cannot be arbitrary, but we must have

$$e^{\pm i 2\pi\alpha} = 1 \quad \Rightarrow \quad \alpha \text{ must be an integer } \alpha = m \in \mathbb{Z}$$

We thus have

$$J|m\rangle = m|m\rangle$$

$$U^m(\varphi)|m\rangle = |m\rangle e^{-im\varphi}$$

We can see that

- $m=0$ corresponds to the trivial representation $U(\varphi) = 1$

- $m=1$ corresponds to $R(\varphi) \to U^1(\varphi) = e^{-i\varphi}$ ↳ isomorphism of the SO(2) group with the unit circle (covered in the clockwise direction)

- $m=-1$ corresponds to $R(\varphi) \to U^{-1}(\varphi) = e^{i\varphi}$ ↳ isomorphism of the SO(2) group with the unit circle (covered in the counterclockwise direction)

  both $m = \pm 1$ representations are FAITHFUL

- $m=2$ corresponds to $R(\varphi) \to U^2(\varphi) = e^{-i2\varphi}$ → the circle is covered twice $\Rightarrow$ not faithful

....

IN SUMMARY: Irreducible representations of $SO(2)$ are given by $J = m \in \mathbb{Z}$ and $U^m(\varphi) = e^{-im\varphi}$

The defining equation of $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is a 2-dimensional representation and thus it must be reducible.

Indeed it is the direct sum of the +1 and −1 representations. To check this we can diagonalize $J = \begin{pmatrix} & -i \\ i & \end{pmatrix}$ and we will obtain two eigenvalues $\lambda = \pm 1$.

ROTATIONS IN 3 DIMENSIONS, SO(3) AND SU(2)

The $SO(3)$ group consists of all continuous linear transformations in 3 dimensions which leave the length of coordinate vectors invariant

$$x'^i = R^i{}_j \, x^j$$

orthogonal matrices

The requirement that $|\underline{x}'|^2 = |\underline{x}|^2$ gives the condition $RR^T = R^T R = E$.

As discussed for $SO(2)$ since physical rotations can be reached continuously from the identity E and $\det E = 1$, we conclude that all rotation matrices must have $\underline{\det R = 1}$. Orthogonal matrices with determinant $\det R = -1$ correspond to rotations combined with space inversion $\left(R = P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \right)$

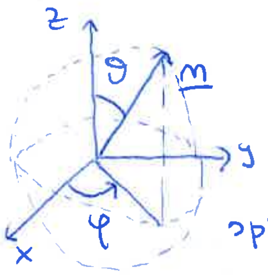
The combination of two rotations is still a rotation and each $SO(3)$ matrix has an inverse $\Rightarrow$ $\underline{SO(3) \text{ is a group}}$.

A rotation in 3 dimensions can be parametrized by <u>3 parameters</u>.

There are infinitely many ways to choose these parameters.

- <u>ANGLE AND AXIS PARAMETRIZATION</u> : $R_{\underline{n}}(\psi)$

A rotation can be fully specified by fixing a unit vector $\underline{n}$ along the axis of the rotation and by an angle ψ. Since $\underline{n}$ is specified by two angles we have three parameters



sphere with unit radius

$$R_{-\underline{n}}(\pi) = R_{\underline{n}}(\pi)$$

redundancy

↳ group parameter space is <u>doubly connected</u> (more later)

$0 \le \psi \le \pi$

$0 \le \vartheta \le \pi$,

$0 \le \varphi < 2\pi$

A useful identity is

$$R_{\underline{n}'}(\psi) = R\, R_{\underline{n}}(\psi)\, R^{-1}$$

where R is a rotation and $\underline{n}' = R\underline{n}$

This identity implies that <u>all rotations by the same angle ψ belong to the same class of SO(3)</u>

Given a fixed rotation axis in the direction $\underline{n}$, rotations about it form a subgroup. This subgroup is isomorphic to SO(2). We can thus introduce a generator $J_{\underline{n}}$ and write

$$R_{\underline{n}}(\psi) = e^{-i\psi J_{\underline{n}}}$$

From the previously seen property of rotations we have

$$R\, J_{\underline{n}}\, R^{-1} = J_{\underline{n}'} \qquad \text{with } \underline{n}' = R\underline{n}$$

Remembering the form of the generator J for $SO(2)$ we can introduce the generators of rotations about the 3 axes $\underline{e}_1, \underline{e}_2, \underline{e}_3$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

this is the structure of the $SO(3)$ generators in the defining representation

We can summarize it by writing

$$(J_k)^{e}{}_{m} = -i\,\varepsilon_{kem}$$

totally antisymmetric rank 3 tensor

One can easily check that the generators in the defining representation satisfy the LIE ALGEBRA

$$[J_i, J_j] = i\,\varepsilon_{ijk} J_k \quad \Rightarrow$$

SO(3) is a non abelian group

One can also show that the generator $J_{\underline{n}}$ of rotations about an arbitrary direction $\underline{n}$ satisfy $J_{\underline{n}} = n_i J_i$

The group commutation rule (Lie algebra) can also be proven by studying the group multiplication close to the identity. We have

$$R_{\underline{n}}(\psi) = e^{-i\psi\, \underline{J}\cdot\underline{n}}$$

and

$$R_2(d\psi)\, J_1\, R_2^{-1}(d\psi) = J_k\, R_2(d\psi)^k{}_1$$

expanding in $d\psi$ we find

$$J_1 - i\,d\psi\,(J_2 J_1 - J_1 J_2) = J_k\left(E^k_1 - i\,d\psi\,(J_2)^k{}_1\right)$$
$$= J_1 - d\psi\, J_3$$

$$\Rightarrow \quad [J_1, J_2] = i J_3$$

We thus see that the Lie algebra $[J_i, J_j] = i\,\varepsilon_{ijk} J_k$ is INDEPENDENT OF THE REPRESENTATION ! It uniquely identifies the local properties of the group.

If on the space of generators one defines the "multiplication" of two generators by taking their commutator, the system forms a linear algebra. This is the reason to use the terminology LIE ALGEBRA.

In physics the generators acquire an even higher importance

$$[H, R_{\underline{n}}(\varphi)] = 0 \qquad \forall \varphi, \underline{n}$$

invariance of the Hamiltonian under rotations (space isotropy)

This implies that

$$[H, J_i] = 0 \qquad i = 1,2,3$$

U unitary $\rightarrow$ J hermitian $\rightarrow$ it corresponds to a physical quantity

What is the quantity whose conservation stems from space isotropy?

$\Rightarrow$ ANGULAR MOMENTUM

Indeed the J_i operators correspond to angular momentum

$$\left(\text{CF : MMP 1} \qquad L_3 = L_z = -i\hbar \frac{\partial}{\partial \varphi} \quad \cdots \right)$$

IRREDUCIBLE REPRESENTATIONS OF SO(3) LIE ALGEBRA

The generators of SO(3) do not commute with each other. However $J^2 = J_1^2 + J_2^2 + J_3^2$ is such that

$$[J^2, J_i] = 0 \qquad i = 1,2,3$$

this relation is easily verified by using $[J_i, J_j] = i\,\varepsilon_{ijk} J_k$.
An operator which commutes with all the elements of a Lie group is called a CASIMIR of that group. By using Schur's lemma, we can say that in any irreducible representation we must have that J^2 is mapped to a multiple of the identity. In other words, in the vector space of a given irreducible representation all vectors are eigenvectors of J^2 with the same eigenvalue. By convention, one chooses the eigenvectors of the commuting operators J^2 and J_3. Defining $J_\pm = J_1 \pm i J_2$ we have

$$[J_3, J_\pm] = \pm J_\pm$$

$J_+^\dagger = J_-$

raising and lowering operators

$$[J_+, J_-] = 2 J_3$$

$$J^2 = J_3^2 + J_3 + J_- J_+ = J_3^2 - J_3 + J_+ J_-$$

Let $|m\rangle$ be a (normalized) eigenvector of J_3 in the space V where m is the eigenvalue (real because J_3 is hermitian)

$$J_3 |m\rangle = m |m\rangle \qquad m \in \mathbb{R}$$

We apply J_+ on $|m\rangle$. The new vector $J_+|m\rangle$ satisfies

$$J_3 J_+|m\rangle = [J_3, J_+]|m\rangle + J_+ J_3|m\rangle$$

$$= J_+|m\rangle + m J_+|m\rangle = (m+1) J_+|m\rangle$$

$\Rightarrow$ $J_+|m\rangle$ is also an eigenvector of J_3 but with eigenvalue $m+1$, or it is the null vector. Similarly, we can show that $J_-|m\rangle$ either vanishes, or it is an eigenvector of J_3 with eigenvalue $m-1$

If $J_+|m\rangle \neq 0$ we can call this vector $|m+1\rangle$ (after appropriate normalization)

Repeating the procedure we get $|m+1\rangle$, $|m+2\rangle$ .....

We require the sequence at some point to terminate, such that V is finite dimensional. Let us call j the maximum. We have

$$J_3|j\rangle = j|j\rangle$$

$$J_+|j\rangle = 0$$

We can thus say that

$$J^2|j\rangle = (J_3^2 + J_3 + J_-J_+)|j\rangle$$

$$= (J_3^2 + J_3)|j\rangle = j(j+1)|j\rangle$$

So, the maximum value of m, j, is related to the eigenvalue of J^2 in the given irreducible representation.

Let us now repeatedly act with J_- on $|j\rangle$. We obtain (after normalization) vectors $|j-1\rangle$, $|j-2\rangle$ ... until the series terminates.

Let us call ℓ the value for which $J_-|\ell\rangle = 0$. We have

$$0 = \langle \ell | J_-^{+} J_- | \ell \rangle = \langle \ell | J_+ J_- | \ell \rangle = \langle \ell | J^2 - J_3^2 + J_3 | \ell \rangle$$

$$= j(j+1) - \ell(\ell-1) \qquad \Rightarrow \quad \ell = -j$$

But $|\ell\rangle = |-j\rangle$ is obtained from $|j\rangle$ by applying J_- an integer number of times $\Rightarrow$ $j-(-j) = 2j$ must be an integer

$\Rightarrow$ $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

The number of vectors we have constructed is $2j+1 = n_j$ ↗ dimension of representation j

We can summarize these results as follows

The irreducible representations of the $SO(3)$ Lie algebra are characterized by an angular momentum eigenvalue j which can be integer or half integer, and by a further (semi-)integer which identifies the basis vectors in the vector space, through the following relations

$$J^2 |jm\rangle = j(j+1) |jm\rangle \qquad J_3 |jm\rangle = m |jm\rangle$$

We also have

$$J_\pm |jm\rangle = |j\, m\pm1\rangle \sqrt{j(j+1) - m(m\pm1)}$$

Indeed let us define λ_m as

$$\langle j\, m+1 | J_+ | jm \rangle = \lambda_m \qquad \left(\text{remember that } (A^{+})_{ij} = A^*_{ji}\right)$$

$$\langle jm | J_- | j\, m+1 \rangle = \lambda_m^* \qquad \left(\text{since } J_+^{+} = J_-\right)$$

$$\langle jm | J_+ J_- | jm\rangle = \langle jm | J_+ | j\, m-1\rangle \langle j\, m-1 | J_- | jm\rangle$$

$$= \lambda_{m-1} \lambda^*_{m-1} = \langle jm | J^2 - J_3^2 + J_3 | jm\rangle$$

$$= j(j+1) - m^2 + m = j(j+1) - m(m-1)$$

$$\Rightarrow \quad \lambda_m = \sqrt{j(j+1) - m(m+1)}$$ note that this holds up to a phase

EXAMPLE

$j = \frac{1}{2} \quad \Rightarrow \quad n_j = 2$

$$J_3 = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix} \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

spinors

we have $J_k = \dfrac{\sigma_k}{2}$ $\quad k = 1,2,3 \quad$ σ_k Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

EXAMPLE

$j = 1 \quad \Rightarrow \quad n_j = 3$ $\qquad \langle 11 | J_+ | 10\rangle = \sqrt{2}$

$$J_3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

This representation is equivalent to the defining representation of $SO(3)$ and it is faithful.

Characters of SO(3) representations

We have seen that rotations by the same angle φ belong to the same class $\Rightarrow$ to compute the character in a given representation we can consider a simple rotation around the z axis. In this case the matrix is diagonal

$$R_3(\varphi) \to \begin{pmatrix} e^{-ij\varphi} & & & \\ & e^{-i(j-1)\varphi} & & \\ & & \ddots & \\ & & & e^{ij\varphi} \end{pmatrix}$$

so the character takes the form

$$\chi^j(\varphi) = \sum_{m=-j}^{j} e^{-im\varphi} = \frac{\sin\left(j+\frac{1}{2}\right)\varphi}{\sin \varphi/2}$$

How about group character orthonormality?

With finite groups we write the orthonormality as $\sum_i \chi_i^{\mu} \chi_i^{\nu}$

Here we have infinite classes! $\leftarrow$ sum over classes

$\Rightarrow$ we have to replace the sum with an integral over an appropriate "measure". The condition takes the form

$$\int_0^{\pi} \underbrace{d\varphi(1-\cos\varphi)}_{\text{group measure}} \chi^j(\varphi)\chi^{j'}(\varphi) = \pi\,\delta_{jj'}$$

Suppose we have a reducible representation with character $\chi(\varphi)$.

The number of times the irreducible representation j occurs in the reduction process is

$$a_j = \frac{1}{\pi} \int_0^\pi d\varphi \, (1-\cos\varphi) \, \chi(\varphi) \, \chi^j(\varphi) .$$

EXAMPLE

$1 \times 1 = 0 + 1 + 2$ let's check it

$$\chi^{1\times 1} = (\chi^1)^2 = \frac{\sin^2 \frac{3}{2}\varphi}{\sin^2 \varphi/2}$$

$$a_0 = \frac{1}{\pi} \int_0^\pi (1-\cos\varphi) \frac{\sin^2 \frac{3}{2}\varphi}{\sin^2 \varphi/2} d\varphi = 1$$

$$a_1 = \frac{1}{\pi} \int_0^\pi (1-\cos\varphi) \frac{\sin^3 \frac{3}{2}\varphi}{\sin^3 \frac{\varphi}{2}} d\varphi = 1$$

$$a_2 = \frac{1}{\pi} \int_0^\pi (1-\cos\varphi) \frac{\sin^2 \frac{3}{2}\varphi}{\sin^2 \frac{\varphi}{2}} \frac{\sin \frac{5}{2}\varphi}{\sin \frac{\varphi}{2}} = 1$$

dimension counting matches!

$m_{1\times 1} = 3 \times 3 = 9$

$m_0 = 1 \qquad m_1 = 3 \qquad m_2 = 5 \qquad$ V

SO(3) AND SU(2)

We have discussed in detail the representations of SO(3). More precisely, we have studied the irreducible representations of the SO(3) algebra, by omitting one important point. For j semi-integer the SO(3) representations are double valued!

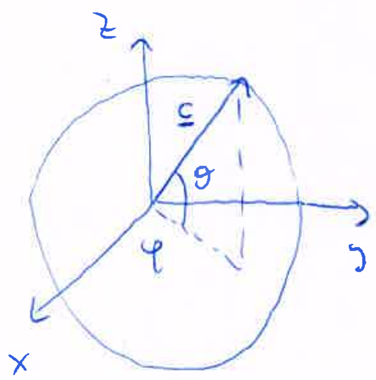
Indeed $\quad D^j\left(R_3(2\pi)\right)^{m'}{}_m = D^j\left[e^{-i2\pi J_3}\right]^{m'}{}_m = \delta_{mm'}\, e^{-i2\pi m}$

but $\quad e^{-i2\pi m} = e^{-i2\pi j}$ since m and j differ by an integer

$$\Rightarrow \quad D^j\left(R_3(2\pi)\right)^{m'}{}_m = \delta_{mm'}\,(-1)^{2j}$$

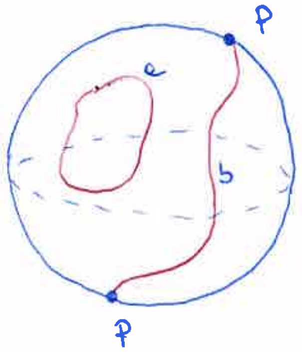
Since $R_3(2\pi) = E$, this means that for j semiinteger E is mapped into $-E$!

The problem is that SO(3) is DOUBLY CONNECTED. To see this let us go back to the angle axis parametrization, in which $R = R_{\underline{n}}(\psi)$. The structure of the group parameter space can be visualized by associating each rotation to a vector $\underline{c} = \psi\,\underline{n}$. The tips of these vectors fill a sphere of radius π.



But $R_{-\underline{n}}(\pi) = R_{\underline{n}}(\pi)$, so two points on the opposite side are equivalent and should be identified!

There are two classes of closed curves on the SO(3) manifold :



a : curves that can be continously deformed to a point

b : curves that cannot be continously deformed to a point

So the SO(3) manifold is COMPACT (because it is closed and bounded) and DOUBLY CONNECTED. We will now see that $SU(2) \sim SO(3)$ but it is SIMPLY CONNECTED (so, to some extent, "easier").

THE SU(2) GROUP

SU(2) is the group of 2x2 unitary matrices with unit determinant.

4 complex parameters $\rightarrow$ 8 real

4 constraints from unitarity condition - 1 from $\det U = 1$

$\Rightarrow$ 3 independent parameters (like SO(3))

If we write
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$$

we have
$$AA^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ ca^* + db^* & cc^* + dd^* \end{pmatrix} = E$$

$$\Rightarrow \begin{cases} |a|^2 + |b|^2 = 1 \\ |c|^2 + |d|^2 = 1 \\ ac^* + bd^* = 0 \end{cases}$$

To study the group manifold a very useful parametrization is

$$A = \begin{pmatrix} z_0 - iz_3 & -z_2 - iz_1 \\ z_2 - iz_1 & z_0 + iz_3 \end{pmatrix} \qquad \text{with } \det A = z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1$$

$z_i \in \mathbb{R}$

$\Rightarrow$ the manifold is the surface of the unit sphere in a 4-dimensional Euclidean space $\rightarrow$ compact and simply connected

The identity corresponds to $z_0 = 1$ and $z_i = 0$ $i = 1,2,3$.

Let us consider an $SU(2)$ matrix close to the identity. We can use

$z_i \rightarrow dz_i$ $i=1,2,3$ and $z_0 = 1 + O(dz_i^2)$ terms

$$\Rightarrow \quad A = \begin{pmatrix} 1 - i\,dz_3 & -dz_2 - i\,dz_1 \\ dz_2 - i\,dz_1 & 1 + i\,dz_3 \end{pmatrix} = E - i\,\sigma_i\,dz_i$$

where σ_i are the Pauli matrices. Since $[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$

we conclude that the Lie algebra of $SU(2)$ is the same of $SO(3)$

with the identification $J_i \sim \sigma_i/2$

$\Rightarrow$ the irreducible representations of the $SO(3)$ algebra are also (single valued) representations of $SU(2)$

The relation between $SU(2)$ and $SO(3)$ can be elucidated as follows.

Let us associate to every $\underline{x} = (x_1, x_2, x_3)$ an hermitian matrix $X = x_i \sigma_i$

We have

$$-\det X = -\begin{vmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{vmatrix} = |\underline{x}|^2$$

X is hermitian and traceless

Now let us take $A \in SU(2)$ and consider the following mapping

$$X \to X' = A X A^{-1}$$

Since $X^+ = X$ and $A \in SU(2)$ we have $(X')^+ = X'$ and $\text{Tr}\, X' = 0$

$\Rightarrow$ X' can be associated to a new coordinate vector $\underline{x}'$ with $|\underline{x}'| = |\underline{x}|$

Therefore the $SU(2)$ transformation induces an $SO(3)$ transformation. But the mapping from $A \in SU(2)$ to $R \in SO(3)$ is $2 \to 1$, since the two matrices $\pm A$ correspond to the same rotation!

In summary:

- $SO(3)$ and $SU(2)$ share the same Lie algebra but they are GLOBALLY different
- Single valued irreducible representations of $SO(3)$ are characterised by INTEGER values of j
- Single valued irreducible representations of $SU(2)$ are labelled by (SEMI)-INTEGER values of j

Spin A particle is said to possess INTRINSIC SPIN s if the quantum mechanical states of the particle at rest are eigenstates of J^2 with eigenvalue $s(s+1)$.

Spin $\frac{1}{2}$ particles transform under the $j = \frac{1}{2}$ representation of $SU(2)$.

CLASSIFICATION OF PHYSICAL QUANTITIES UNDER O(3) REPRESENTATIONS

$$O(3) = \underset{\substack{\downarrow \\ R\ (\text{rotations})}}{SO(3)} \otimes \underset{\substack{\searrow \\ P\ (\text{parity})}}{Z_2} \qquad Z_2 \sim C_2$$

	1	-1
U_+	1	1
U_-	1	-1

- SCALAR $\quad s \xrightarrow{R} s \quad s \xrightarrow{P} s \quad \underset{\substack{\nearrow \\ SO(3)}}{U_0} \otimes \underset{\substack{\uparrow \\ Z_2}}{U_+}$ (mass, charge ...)
- VECTOR $\quad V^i \xrightarrow{R} R^i{}_j V^j \quad V^i \xrightarrow{P} -V^i \quad U_1 \otimes U_-$ (position, velocity, electric field ...)
- PSEUDOSCALAR $\quad \tilde{s} \xrightarrow{R} \tilde{s} \quad \tilde{s} \xrightarrow{P} -\tilde{s} \quad U_0 \otimes U_-$ (flux of magnetic field ...)
- PSEUDOVECTOR $\quad \tilde{V}^i \xrightarrow{R} R^i{}_j \tilde{V}^j \quad \tilde{V}^i \xrightarrow{P} \tilde{V}^i \quad U_1 \otimes U_+$ (magnetic field, angular momentum ...)
- RANK-2 TENSOR $\quad M^{ij} \xrightarrow{R} R^i{}_k R^j{}_l M^{kl} \quad M^{ij} \xrightarrow{P} M^{ij} \quad \underbrace{U_1 \otimes U_1}_{\text{reducible!}} \otimes U_+$

We have already seen that $U_1 \otimes U_1 = U_0 \oplus U_1 \oplus U_2$

Indeed the tensor M^{ij} can be written as

$$M^{ij} = \underbrace{\mathrm{Tr}M\, \delta^{ij}}_{\text{scalar } (j=0)} + \underbrace{\frac{1}{2}(M^{ij} - M^{ji})}_{\text{vector } (j=1)} + \left(\frac{M^{ij} + M^{ji}}{2} - \mathrm{Tr}M\, \delta^{ij} \right)$$

$$\begin{pmatrix} 0 & \times & \times \\ & 0 & \times \\ & & 0 \end{pmatrix} \quad \text{vector } (j=1)$$

symmetric traceless

$\Rightarrow$ $6-1 = 5$ components

$\Rightarrow$ $j=2$

FORM INVARIANCE (COVARIANCE) OF PHYSICAL LAWS

In classical physics physical laws should take the same form in frames connected by a Galilei transformation

- $\underline{F} = m \, \underline{a}$

$\underline{F}$: vector; m: scalar; $\underline{a}$: vector

left and right hand side of the equation transform in the same way

$\Rightarrow$ in a different frame obtained by a $O(3)$ transformation the equation takes the same form!

- $U = -\underline{d} \cdot \underline{E} - \underline{\mu} \cdot \underline{B}$

U: energy (scalar); $\underline{d}$: electric dipole moment (vector); $\underline{E}$: electric field; $\underline{\mu}$: magnetic moment (pseudovector); $\underline{B}$: magnetic field (pseudovector)

Scalar product of two vectors (or two pseudovectors) is a SCALAR

When moving from CLASSICAL MECHANICS to SPECIAL RELATIVITY the situation changes: TIME AND ENERGY are not any more SCALARS and POSITION and MOMENTUM are not any more VECTORS

$\Rightarrow$ 4-VECTORS HAVE TO BE CONSIDERED