GROUP THEORY

Basic concepts

RECALL: A GROUP is a set of elements $g_i \in G$ with

(i) an operation ("multiplication") $G \times G \to G \mid \forall g_1, g_2 \in G \quad g_3 = g_1 \circ g_2 \in G$

and $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (ASSOCIATIVE PROPERTY)

(ii) $\exists\, e \in G$ (NEUTRAL ELEMENT) $\mid g \circ e = e \circ g = g \quad \forall g \in G$

(iii) $\forall g \in G \quad \exists\, g^{-1} \in G \mid g \circ g^{-1} = g^{-1} \circ g = e$

Why studying groups? The concept of group has a number of important applications in physics. Deep relation with SYMMETRIES

EXAMPLE a) $G = \mathbb{R}^+$ i.e. $G = \{ x \in \mathbb{R} \mid x > 0 \}$ $\quad e = 1 \quad x^{-1} = \frac{1}{x}$ ordinary multiplication

EXAMPLE b) $G = \mathbb{R} \quad g_1 \circ g_2 = g_1 + g_2$, $e = 0 \quad g^{-1} = -g$ addition

EXAMPLE c) $n \times n$ matrices g with $\det(g) \neq 0$ $\quad g_1 \circ g_2$ matrix multiplication

$e = I \quad g^{-1}$ inverse matrix

($GL(n, \mathbb{K})$ general linear group)

DEF A group is ABELIAN if $\forall g_1, g_2 \in G \quad g_1 \circ g_2 = g_2 \circ g_1$

(EXAMPLES a) and b) are abelian groups, c) is example of non abelian group

DEF A finite group is a group with a finite number of elements

In this case the ORDER of the group is the number of its elements

- The simplest group consists just of one element $G = \{e\}$. Since $e \cdot e = e$ we have $e = e^{-1}$. The number 1 with the usual multiplication is an example. We denote this group by C_1

- The next-to-simplest group is $G = \{e, a\}$. According to the properties of e we must have $ae = ea = a$, so we need to specify only the product aa. Should we set $aa = a$ or $aa = e$? In the first case by multiplying by a^{-1} we would get $a = e$, so we have to set $aa = e$, that is, $a^{-1} = a$. An example of this group is $G = \{1, -1\}$ with the usual multiplication. We call this group C_2

Examples of this group structure are $G = \{I, P\}$ where P is the parity transformation $\underline{x} \to -\underline{x}$ and I is the identity $\underline{x} \to -\underline{x}$, and the trasposition of a matrix, together with the identity $\left((A^T)^T = A\right)$.

- The next example is the one of the group $C_3 = \{e, a, b\}$ with the multiplication table:

e	a	b
a	b	e
b	e	a

Since $a^{-1} = b$ we can denote the group also as

$$C_3 = \{e, a, \underset{"a^2"}{a^{-1}}\} \quad \text{with the requirement } a^3 = e$$

An example of this group is given by $G = \{1, e^{i 2\pi/3}, e^{-i 2\pi/3}\}$ with the usual multiplication.

The above examples are examples of the CYCLIC GROUPS C_n, which have the general structure $\{e, a, a^2, \dots a^{n-1} ; a^n = e\}$. The rows and columns of their multiplication table are cyclic permutations of each other! The cyclic groups are all abelian.

Up to order $n=3$ the cyclic groups are the only possible groups. Indeed for C_3 we have

e	a	b
a	b	e
b	e	a

suppose that $aa \neq b$

$$\Rightarrow \text{ if } aa = e \Rightarrow ab = \begin{cases} a \Rightarrow b = e \\ b \Rightarrow a = e \\ e \Rightarrow a = b \end{cases}$$

$\Rightarrow$ it does not work!

- The simplest non-cyclic group is of order 4 and is called four-group or dihedral group and denoted by D_2

$G = \{e, a, b, c\}$

e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

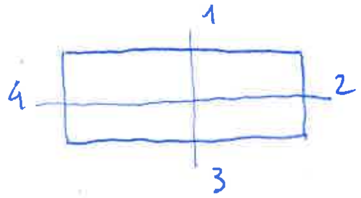
this is still an abelian group

$\downarrow$

symmetry group of a rectangle

Note that C_4 is different!

e	a	b	c
a	b	c	e
b	c	e	a
c	e	a	b

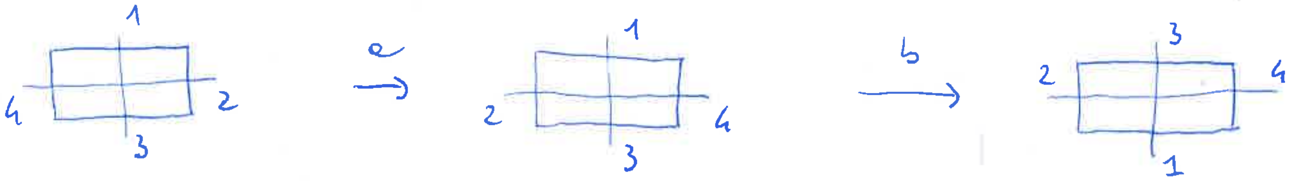


$e =$ leave figure unchanged

$a =$ reflection about the vertical axis

$b =$ " " " horizontal axis

$c =$ rotation around the center by 180°



$\Rightarrow$ we have $ab = c$ $\quad a^2 = e \quad b^2 = e \quad c^2 = e$

SUBGROUPS, CLASSES AND INVARIANT SUBGROUPS

DEF: A subset $H \subset G$ which forms a group under the same multiplication rule of G is called SUBGROUP of G

EXAMPLE: $\mathbb{Q}^+$ is a subgroup of $\mathbb{R}^+$ under multiplication

DEF: A subgroup H of G is called INVARIANT if $\forall h \in H$, $\forall g \in G$ we have $ghg^{-1} \in H$

DEF: An element $b \in G$ is said to be CONJUGATE to $a \in G$ if $\exists\, p \in G \mid b = pap^{-1}$. We write $a \sim b$

It is easy to see that conjugation is an equivalence relation i) $a \sim a$ ii) if $a \sim b \Rightarrow b \sim a$ iii) if $a \sim b$ and $b \sim c \Rightarrow a \sim c$

Elements of a group which are conjugate to each other are said to form a class. If G is abelian, every $h \in G$ forms a separate class, since $ghg^{-1} = h$.

- From these definitions it follows that if $H \subset G$ is an invariant subgroup $\Rightarrow$ H is the union of its conjugate classes $H = \bigcup_{g \in H} Cl(g)$

(this can also be taken as an alternative definition of INVARIANT SUBGROUP)

Indeed, if $h \in H$ and $h' \sim h$, but $h' \notin H$ $\Rightarrow$ $h' = ghg^{-1} \notin H$ and H cannot be an invariant subgroup.

- Every group G has at least two trivial invariant subgroups: $\{e\}$ and G itself.

The group is said to be SIMPLE if it does not contain other non-trivial invariant subgroups. A group is SEMISIMPLE if it does not contain any ABELIAN invariant subgroup.

ISOMORPHISM

Two groups G and G' are said to be isomorphic if there exists a one-to-one correspondence between their elements which is preserved by the multiplication rule. In other words, if $g_i \in G \to g_i' \in G'$ and $g_1 \cdot g_2 = g_3 \Rightarrow g_1' \cdot g_2' = g_3'$

EXAMPLE

The group $G = \{\pm 1, \pm i\}$ with the usual multiplication is isomorphic to C_4 (define $e = 1$, $\underset{a}{a = i}$, $\underset{a^2}{b = -1}$, $\underset{a^3}{c = -i}$)

THE REARRANGEMENT LEMMA AND THE SYMMETRIC GROUP

<u>LEMMA</u> : If $P, b, c \in G$ and $pb = pc \Rightarrow b = c$ (just multiply by p^{-1})

Let us consider the case of a finite group G of order m, and denote the group by $\{g_1, g_2, \cdots g_m\}$. The multiplication by a fixed $h \in G$ introduces a <u>PERMUTATION</u>

$$\{g_1, g_2 \cdots g_m\} \longrightarrow \{hg_1, hg_2 \cdots hg_m\} = \{g_{h_1}, g_{h_2} \cdots g_{h_m}\}$$

where $\{h_1, h_2 \cdots h_m\}$ is a permutation of $\{1, 2, \cdots m\}$ determined by h.

This is because, since the g_i are all different from each other, the multiplication by h also leads to m different group elements hg_i.

- We introduce the group of <u>permutations</u>.

An arbitrary permutation of m objects is denoted by

$$P = \begin{pmatrix} 1 & 2 & \cdots & m \\ P_1 & P_2 & \cdots & P_m \end{pmatrix}$$

each entry in the first row is replaced by an entry in the second row

The set of $m!$ permutations of m objects forms the group $\underline{S_m}$ called the <u>permutation group</u> or <u>symmetric group</u>

$$e = \begin{pmatrix} 1 & 2 & \cdots & m \\ 1 & 2 & \cdots & m \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} P_1 & P_2 & \cdots & P_m \\ 1 & 2 & \cdots & m \end{pmatrix}$$

A more compact notation is based on the CYCLE STRUCTURE

EXAMPLE

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}$$

here 1 is replaced by 3 and 3 is replaced by 4 which is replaced by 1

$\Rightarrow$ we use a 3-cycle notation (134)

Similarly, 2 and 5 give a 2-cycle (25) while 6 is unchanged

$\Rightarrow$ we can write $P = (134)(25)(6)$

The neutral element is $e = (1)(2)(3) \cdots (n)$

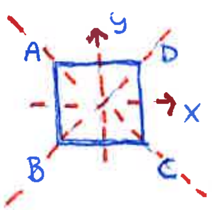
DEF : CLASS OF A PERMUTATION : $(-1)^m$ ↳ number of transpositions with which it can be realised (even or odd)

$\Rightarrow$ P in the previous example is ODD

DIHEDRAL GROUPS D_n

The dihedral group is the group of symmetries of a regular polygon. A regular polygon with n sides has $2n$ symmetries : n reflections and n rotations by multiples of $2\pi/n$ $\Rightarrow$ there are $2n$ elements

example : D_4



8 elements :

e identity

t_x, t_y reflections about the x and y axes

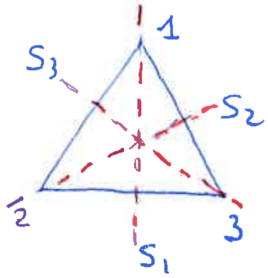
t_{AC}, t_{BD} reflections about the diagonals

r, r^2, r^3 rotations by $30°, 180°, 270°$ about the center

THE GROUP D_3 ($\sim S_3$)

Up to now we have only seen examples of abelian groups. The smallest non-abelian group has 6 elements and is the group D_3 (symmetry transformation of the equilateral triangle). The group is isomorphic to the symmetric group S_3 and we will use both notations.

	S_3 notation
e : $(1,2,3) \to (1,2,3)$	$(1)(2)(3)$
$\frac{2}{3}\pi$ rotation $r_1 : (1,2,3) \to (2,3,1)$	(123)
$\frac{4}{3}\pi$ rotation $r_2 : (1,2,3) \to (3,1,2)$	(132)
reflections $s_1 : (1,2,3) \to (1,3,2)$	$(1)(23)$
$s_2 : (1,2,3) \to (3,2,1)$	$(2)(13)$
$s_3 : (1,2,3) \to (2,1,3)$	$(3)(21)$

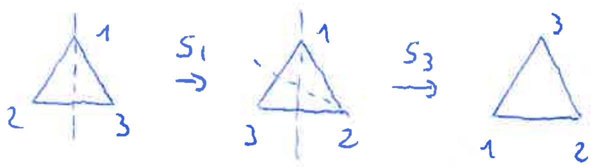


We have :

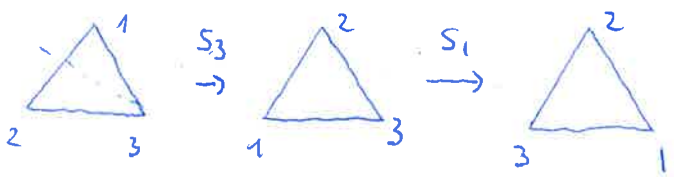
$$r_1 \cdot r_1 = r_2 \qquad r_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad r_1^{-1} = r_2$$

The group is non-abelian !

$s_3 \cdot s_1 = r_1$ indeed



$s_1 \cdot s_3 = r_2$ indeed



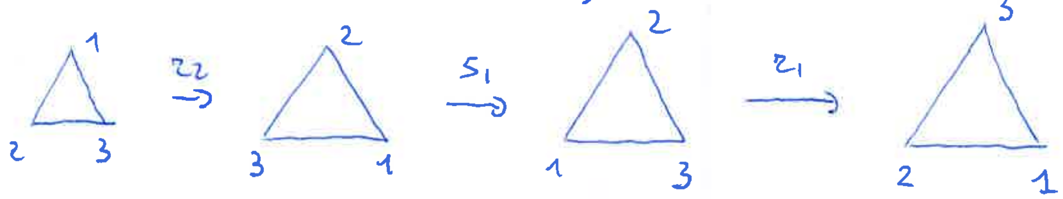
We now analyze the subgroups of D_3. We have the trivial subgroups $\{e\}$ and D_3.

We also have the subgroups

$$\{e, s_1\}\ ,\ \{e, s_2\}\ ,\ \{e, s_3\} \implies \text{all isomorphic to } C_2$$

These subgroups are NOT invariant. Indeed let us evaluate $r_1 s_1 r_1^{-1}$

$$r_1 s_1 r_1^{-1} = r_1 s_1 r_2 = s_2 \notin \{e, s_1\}$$



We can easily check that $\{e, r_1, r_2\}$ is an invariant subgroup (rotations)

$\implies$ D_3 is NOT SIMPLE

We also note that $\{e, s_1, s_2, s_3\}$ is not even a subgroup!

The conjugate classes of D_3 are

$$\begin{array}{cccc}
\{e\} , & \{s_1, s_2, s_3\} , & \{r_1, r_2\} & \\
\downarrow & \downarrow & \downarrow & \\
(\cdot)(\cdot)(\cdot) & (\cdot)(\cdot\cdot) & (\cdot\cdot\cdot) & S_3 \text{ notation}
\end{array}$$

We have seen that D_3 is of order 6 and that it is isomorphic to S_3 which is a subgroup of S_6.

This is a special case of a general theorem due to CAYLEY.

CAYLEY THEOREM

Every finite group of order n is isomorphic to a subgroup of S_n

PROOF

$$a \in G \quad \to \quad P_a = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & & a_n \end{pmatrix} \in S_n$$

indices determined by the defining identity

$$g_{a_i} = a g_i \qquad i = 1, 2 \cdots n$$

Suppose $ab = c$ in G

$$P_a P_b = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n \\ b_1 & \cdots & & b_n \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ a_{b_1} & a_{b_2} & \cdots & a_{b_n} \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & \cdots & n \\ b_1 & b_2 & & b_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \cdots & n \\ a_{b_1} & a_{b_2} & & a_{b_n} \end{pmatrix}$$

but $g_{a_{b_i}} = a g_{b_i} = a(b g_i) = (ab) g_i = c g_i$

$$\Rightarrow \quad P_a P_b = \begin{pmatrix} 1 & 2 & \cdots & n \\ c_1 & c_2 & & c_n \end{pmatrix} \quad \Rightarrow \quad ab = c \text{ in } G \text{ implies } P_a \cdot P_b = P_c \text{ in } S_n$$

This means that the mapping $a \in G \to P_a \in S_n$ preserves multiplication

$\Rightarrow$ The permutations $P_a = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & & a_n \end{pmatrix}$ for all $a \in G$ form a subgroup of S_n which is isomorphic to G ∎

EXAMPLE

Let us consider the group $C_4 = \{e, a, b, c\}$ is isomorphic to a subgroup of S_4

$$H = \{e, (1234), (13)(24), (4321)\} \leftrightarrow \{e, a, b, c\}$$

Indeed we have

$$\overset{a}{(1234)} \cdot \overset{b}{\left[(13)(24)\right]} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \overset{c}{(4321)}$$

$$\overset{a}{(1234)} \cdot \overset{a}{(1234)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \overset{b}{(13)(24)}$$

e	a	b	c
a	b	c	e
b	c	e	a
c	e	a	b

SUMMARY ON FINITE GROUPS

C_m $\{e, a, a^2 \dots a^{m-1}\}$ $a^m = e$ abelian group of order $\underline{m}$

S_m group of permutations of m elements → order is $\underline{m!}$

D_m dihedral group : symmetry group of regular polygons with m sides → order $\underline{2m}$

SOME CONTINOUS GROUPS

$O(m)$ orthogonal matrices $m \times m$ $AA^T = A^TA = I$ $\det A = \pm 1$

transformations that preserve the length of m-component vectors (rotations + reflections)

$SO(m)$ $A \in O(m) \mid \det A = 1$ (rotations)

$U(m)$ unitary $m \times m$ complex matrices $U^\dagger U = UU^\dagger = I$

$SU(m)$ " " " with $\det U = 1$

SPACE-TIME SYMMETRIES

- translation $\underline{x} \to \underline{x} + a$ (space) $t \to t + t_0$ (time) — HOMOGENEITY OF SPACE AND TIME
- rotation $\underline{x} \to R\,\underline{x}$ — ISOTROPY OF SPACE
- Lorentz transformation
  ↳ extends the classical spacetime symmetries → SPECIAL RELATIVITY